Lane CC Math 65 – Elementary Algebra

An open source (CC-BY) textbook by Tyler Wallace

Edited for Lane CC by Berri Hsiao and Wendy Rawlinson

Beginning and Intermediate Algebra

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by Tyler Wallace

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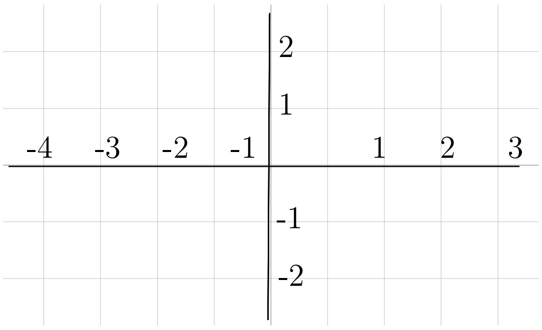
Chapter 0 - Graphing

0.1 Points and Lines

Objective: Graph points and lines using xy coordinates.

Often, to get an idea of the behavior of an equation we will make a picture that represents the solutions to the equations. A **graph** is simply a picture of the solutions to an equation. Before we spend much time on making a visual representation of an equation, we first have to understand the basis of graphing. Following is an example of what is called the coordinate plane.



The plane is divided into four sections by a horizontal number line (x-axis) and a vertical number line (y-axis). Where the two lines meet in the center is called the origin. This center origin is where $x=0$ and $y=0$. As we move to the right the numbers count up from zero, representing $x=1,2,3\ldots$.

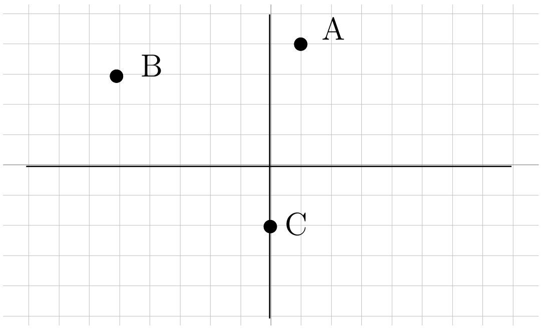
To the left the numbers count down from zero, representing $x=-1,-2,-3$. Similarly, as we move up the number count up from zero, $y=1,2,3$., and as we move down count down from zero, $y=-1,-2,-3$. We can put dots on the graph which we will call points. Each point has an "address" that defines its location. The first number will be the value on the $x-$axis or horizontal number line. This is the distance the point moves left/right from the origin. The second number will represent the value on the $y-$ axis or vertical number line. This is the distance the point moves up/down from the origin. The points are given as an ordered pair (x,y).

World View Note: Locations on the globe are given in the same manner, each number is a distance from a central point, the origin which is where the prime meridian and the equator. This "origin is just off the western coast of Africa.

The following example finds the address or coordinate pair for each of several points on the coordinate plane.

Example 119.

Give the coordinates of each point.



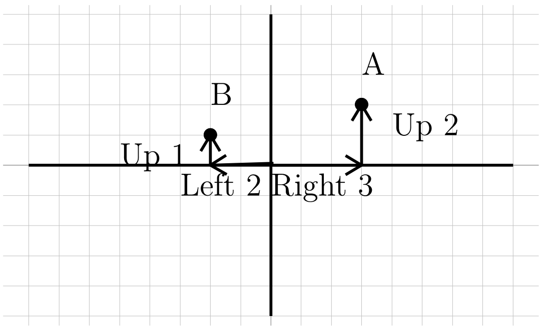
Tracing from the origin, point A is right 1, up 4. This becomes A(1, 4). Point B is left 5, up 3. Left is backwards or negative so we have B(− 5, 3). C is straight down 2 units. There is no left or right. This means we go right zero so the point is C(0, − 2).

$$A(1,4), B(-5,3), C(0,-2) \qquad \text{Our Solution}$$

Just as we can give the coordinates for a set of points, we can take a set of points and plot them on the plane.

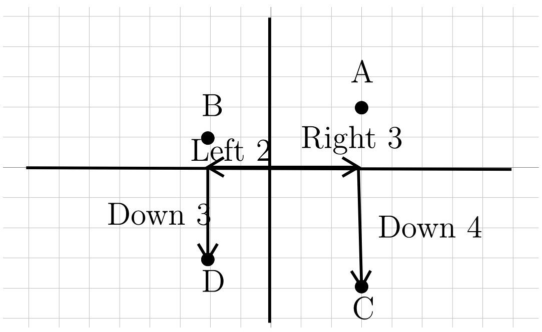
Example 120.

Graph the points $A(3,2)$, $B(-2,1)$, $C(3,-4)$, $D(-2,-3)$, $E(-3,0)$, $F(0,2)$, $G(0,0)$



The first point, A is at (3, 2) this means $x=3$ (right 3) and $y=2$ (up 2). Following these instructions, starting from the origin, we get our point.

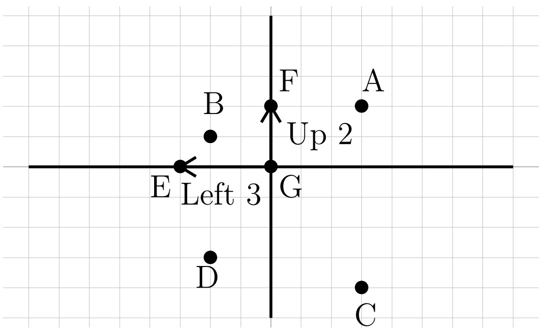
The second point, $B(-2,1)$, is left 2 (negative moves backwards), up 1. This is also illustrated on the graph.



The third point, $C(3,-4)$ is right 3, down 4 (negative moves backwards).

The fourth point, D $(-2,-3)$ is left 2, down 3 (both negative, both move backwards)

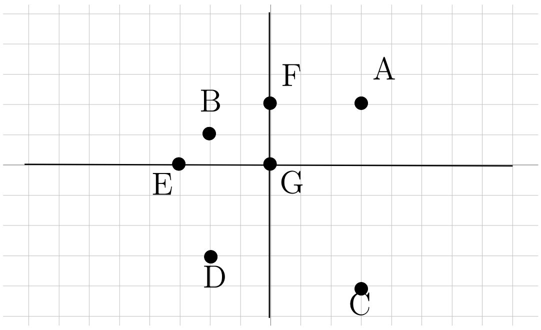
The last three points have zeros in them. We still treat these points just like the other points. If there is a zero there is just no movement.



Next is $E(-3,0)$. This is left 3 (negative is backwards), and up zero, right on the $x-$axis.

Then is $F(0,2)$. This is right zero, and up two, right on the $y-$axis.

Finally is $G(0,0)$. This point has no movement. Thus the point is right on the origin.



Our Solution

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $y = 2x - 3$. We may be interested in what type of solution are possible in this equation. We can visualize the solution by making a graph of possible x and y combinations that make this equation a true statement. We will have to start by finding possible x and y combinations. We will do this using a table of values.

Example 121.

Graph $y = 2x - 3$ We make a table of values

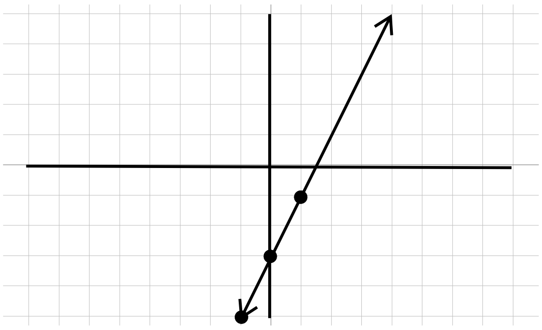
x	y
-1	
0	
1	

We will test three values for x. Any three can be used

x	y
-1	-5
0	-3
1	-1

Evaluate each by replacing x with the given value
$x = -1; y = 2(-1) - 3 = -2 - 3 = -5$
$x = 0; y = 2(0) - 3 = 0 - 3 = -3$
$x = 1; y = 2(1) - 3 = 2 - 3 = -1$

$(-1, -5), (0, -3), (1, -1)$ These then become the points to graph on our equation



Plot each point.

Once the point are on the graph, connect the dots to make a line.

The graph is our solution.-

What this line tells us is that any point on the line will work in the equation $y = 2x - 3$. For example, notice the graph also goes through the point $(2, 1)$. If we use $x = 2$, we should get $y = 1$. Sure enough, $y = 2(2) - 3 = 4 - 3 = 1$, just as the graph suggests. Thus we have the line is a picture of all the solutions for $y = 2x - 3$. We can use this table of values method to draw a graph of any linear equation.

Example 122.

Graph $2x - 3y = 6$ — We will use a table of values

x	y
-3	
0	
3	

We will test three values for x. Any three can be used.

$$\begin{array}{rl}
2(-3) - 3y = 6 & \text{Substitute each value in for } x \text{ and solve for } y\\
-6 - 3y = 6 & \text{Start with } x = -3\text{, multiply first}\\
\underline{+6 \qquad +6} & \text{Add 6 to both sides}\\
-3y = 12 & \text{Divide both sides by } -3\\
\overline{-3} \quad \overline{-3} & \\
y = -4 & \text{Solution for } y \text{ when } x = -3\text{, add this to table}
\end{array}$$

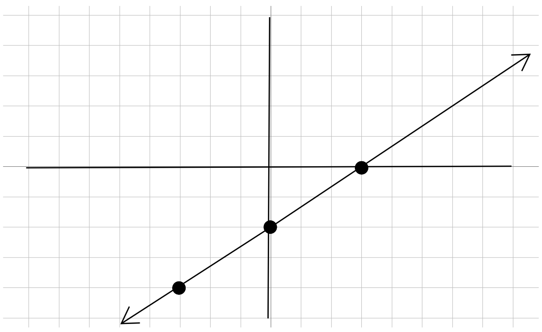
$$\begin{array}{rl}
2(0) - 3y = 6 & \text{Next } x = 0\\
-3y = 6 & \text{Multiplying clears the constant term}\\
\overline{-3} \; \overline{-3} & \text{Divide each side by } -3\\
y = -2 & \text{Solution for } y \text{ when } x = 0\text{, add this to table}
\end{array}$$

$$\begin{array}{rl}
2(3) - 3y = 6 & \text{Next } x = 3\\
6 - 3y = 6 & \text{Multiply}\\
\underline{-6 \qquad -6} & \text{Subtract 9 from both sides}\\
-3y = 0 & \text{Divide each side by } -3\\
\overline{-3} \quad \overline{-3} & \\
y = 0 & \text{Solution for } y \text{ when } x = -3\text{, add this to table}
\end{array}$$

x	y
-3	-4
0	-2
3	0

Our completed table.

$(-3,-4),(0,2),(3,0)$ Table becomes points to graph

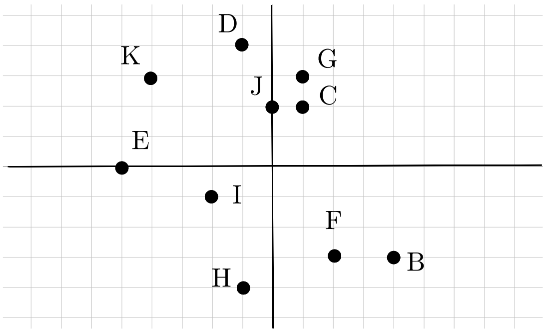


Graph points and connect dots

Our Solution

0.1 Practice - Points and Lines

State the coordinates of each point.

1) 

Plot each point.

2) L$(-5,5)$ K$(1,0)$ $J(-3,4)$

I$(-3,0)$ $H(-4,2)$ G(4, -2)

$F(-2,-2)$ E $(3,-2)$ D(0, 3)

C (0, 4)

Sketch the graph of each line.

3) $y=-\frac{1}{4}x-3$

4) $y=x-1$

5) $y=-\frac{5}{4}x-4$

6) $y=-\frac{3}{5}x+1$

7) $y=-4x+2$

8) $y=\frac{5}{3}x+4$

9) $y=\frac{3}{2}x-5$

10) $y=-x-2$

11) $y=-\frac{4}{5}x-3$

12) $y=\frac{1}{2}x$

13) $x+5y=-15$

14) $8x-y=5$

15) $4x+y=5$

16) $3x+4y=16$

17) $2x-y=2$

18) $7x+3y=-12$

19) $x+y=-1$

20) $3x+4y=8$

21) $x-y=-3$

22) $9x-y=-4$

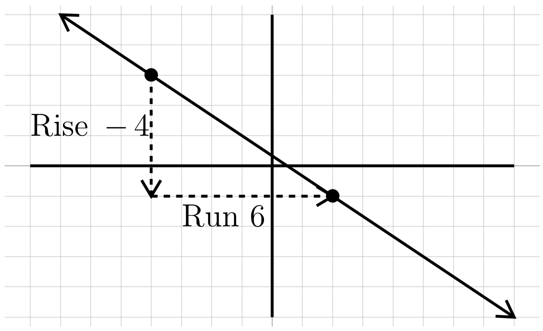
0.2 Slope

Objective: Find the slope of a line given a graph or two points.

As we graph lines, we will want to be able to identify different properties of the lines we graph. One of the most important properties of a line is its slope. **Slope** is a measure of steepness. A line with a large slope, such as 25, is very steep. A line with a small slope, such as $\frac{1}{10}$ is very flat. We will also use slope to describe the direction of the line. A line that goes up from left to right will have a positive slope and a line that goes down from left to right will have a negative slope.

As we measure steepness we are interested in how fast the line rises compared to how far the line runs. For this reason we will describe slope as the fraction $\frac{\text{rise}}{\text{run}}$. Rise would be a vertical change, or a change in the y-values. Run would be a horizontal change, or a change in the x-values. So another way to describe slope would be the fraction $\frac{\text{change in } y}{\text{change in } x}$. It turns out that if we have a graph we can draw vertical and horiztonal lines from one point to another to make what is called a slope triangle. The sides of the slope triangle give us our slope. The following examples show graphs that we find the slope of using this idea.

Example 123.

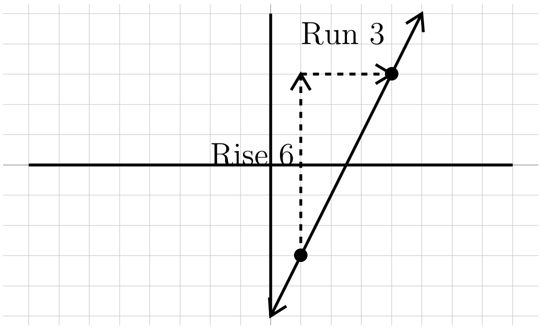


To find the slope of this line we will consider the rise, or verticle change and the run or horizontal change. Drawing these lines in makes a slope triangle that we can use to count from one point to the next the graph goes down 4, right 6. This is rise -4, run 6. As a fraction it would be, $\frac{-4}{6}$. Reduce the fraction to get $-\frac{2}{3}$.

$$-\frac{2}{3} \quad \text{Our Solution}$$

World View Note: When French mathematicians Rene Descartes and Pierre de Fermat first developed the coordinate plane and the idea of graphing lines (and other functions) the y-axis was not a vertical line!

Example 124.

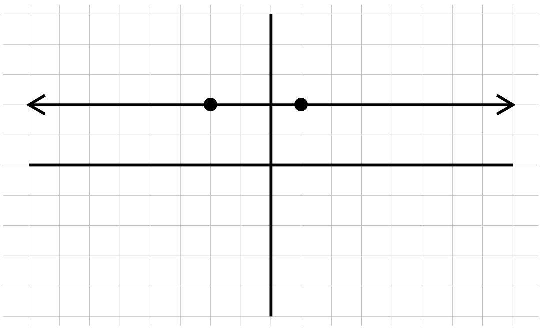


To find the slope of this line, the rise is up 6, the run is right 3. Our slope is then written as a fraction, $\frac{\text{rise}}{\text{run}}$ or $\frac{6}{3}$. This fraction reduces to 2. This will be our slope.

2 Our Solution

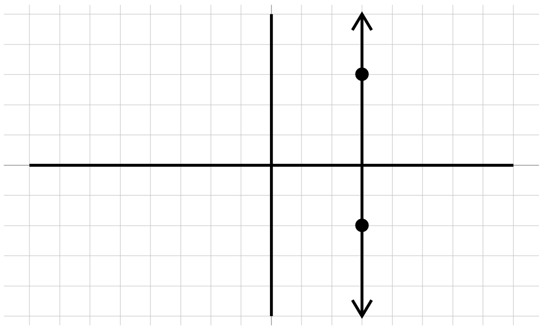
There are two special lines that have unique slopes that we need to be aware of. They are illustrated in the following example.

Example 125.



In this graph there is no rise, but the run is 3 units. This slope becomes

$\frac{0}{3} = 0$. This line, and all horizontal lines have *a* zero slope.



This line has a rise of 5, but no run. The slope becomes $\frac{5}{0} =$ undefined. This line, and all vertical lines, have no slope.

As you can see there is a big difference between having a zero slope and having no slope or undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all, in fact it is flat. Therefore it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express how steep it is. This is an undefined slope.

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y values, we can calculate this by subtracting the y values of a point. Similarly, if run is a change in the x values, we can calculate this by subtracting the x values of a point. In this way we get the following equation for slope.

$$\textbf{The slope of } \boldsymbol{a} \textbf{ line through } (\boldsymbol{x_1}, \boldsymbol{y_1}) \textbf{ and } (\boldsymbol{x_2}, \boldsymbol{y_2}) \textbf{ is } \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}}$$

When mathematicians began working with slope, it was called the modular slope. For this reason we often represent the slope with the variable m. Now we have the following for slope.

$$\textbf{Slope} = \boldsymbol{m} = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } \boldsymbol{y}}{\textbf{change in } \boldsymbol{x}} = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}}$$

As we subtract the y values and the x values when calculating slope it is important we subtract them in the same order. This process is shown in the following examples.

Example 126.

Find the slope between $(-4,3)$ and $(2,-9)$	Identify x_1, y_1, x_2, y_2
(x_1, y_1) and (x_2, y_2)	Use slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$
$m = \frac{-9-3}{2-(-4)}$	Simplify
$m = \frac{-12}{6}$	Reduce
$m = -2$	Our Solution

Example 127.

Find the slope between $(4, \quad 6)$ and $(2,-1)$	Identify x_1, y_1, x_2, y_2
(x_1, y_1) and (x_2, y_2)	Use slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$
$m = \frac{-1-6}{2-4}$	Simplify
$m = \frac{-7}{-2}$	Reduce, dividing by -1
$m = \frac{7}{2}$	Our Solution

We may come up against a problem that has a zero slope (horiztonal line) or no slope (vertical line) just as with using the graphs.

Example 128.

Find the slope between $(-4,-1)$ and $(-4,-5)$	Identify x_1, y_1, x_2, y_2

$$(x_1\ , y_1) \text{ and } \ (x_2,\ y_2) \quad \text{Use slope formula, } m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-5-(-1)}{-4-(-4)} \quad \text{Simplify}$$

$$m = \frac{-4}{0} \quad \text{Can}'t \text{ divide by zero, undefined}$$

$$m = \text{no slope} \quad \text{Our Solution}$$

Example 129.

$$\text{Find the slope between } (3,1) \text{ and } (-2,1) \quad \text{Identify } x_1, y_1, x_2, y_2$$

$$(x_1, y_1) \text{ and } (x_2, y_2) \quad \text{Use slope formula, } m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{1-1}{-2-3} \quad \text{Simplify}$$

$$m = \frac{0}{-5} \quad \text{Reduce}$$

$$m = 0 \quad \text{Our Solution}$$

Again, there is a big difference between no slope and a zero slope. Zero is an integer and it has a value, the slope of a flat horizontal line. No slope has no value, it is undefined, the slope of a vertical line.

Using the slope formula we can also find missing points if we know what the slope is. This is shown in the following two examples.

Example 130.

Find the value of y between the points $(2, y)$ and $(5, -1)$ with slope -3

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{We will plug values into slope formula}$$

$$-3 = \frac{-1-y}{5-2} \quad \text{Simplify}$$

$$-3 = \frac{-1-y}{3} \quad \text{Multiply both sides by 3}$$

$$-3(3) = \frac{-1-y}{3}(3) \quad \text{Simplify}$$

$$-9 = -1 - y \quad \text{Add 1 to both sides}$$

$$\underline{+1 \quad +1}$$

$$-8 = -y \quad \text{Divide both sides by } -1$$

$$\overline{-1} \quad \overline{-1}$$

$$8 = y \quad \text{Our Solution}$$

Example 131.

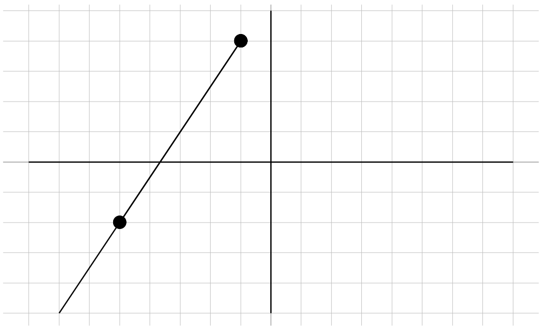
Find the value of x between the points $(-3, 2)$ and $(x, 6)$ with slope $\frac{2}{5}$

$$
\begin{array}{rl}
m = \dfrac{y_2 - y_1}{x_2 - x_1} & \text{We will plug values into slope formula} \\
\dfrac{2}{5} = \dfrac{6 - 2}{x - (-3)} & \text{Simplify} \\
\dfrac{2}{5} = \dfrac{4}{x + 3} & \text{Multiply both sides by } (x+3) \\
\dfrac{2}{5}(x + 3) = 4 & \text{Multiply by 5 to clear fraction} \\
(5)\dfrac{2}{5}(x + 3) = 4(5) & \text{Simplify} \\
2(x + 3) = 20 & \text{Distribute} \\
2x + 6 = 20 & \text{Solve.} \\
\underline{-6 \quad -6} & \text{Subtract 6 from both sides} \\
2x = 14 & \text{Divide each side by 2} \\
\overline{\ 2\ } \quad \overline{\ 2\ } & \\
x = 7 & \text{Our Solution}
\end{array}
$$

0.2 Practice - Slope

Find the slope of each line.

1)



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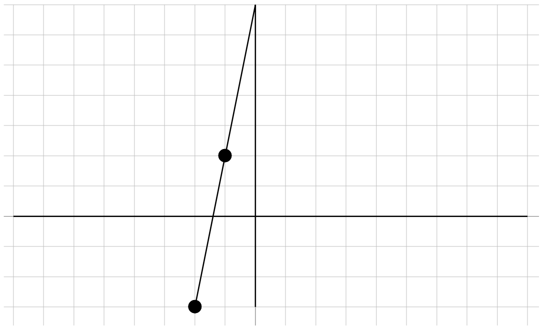

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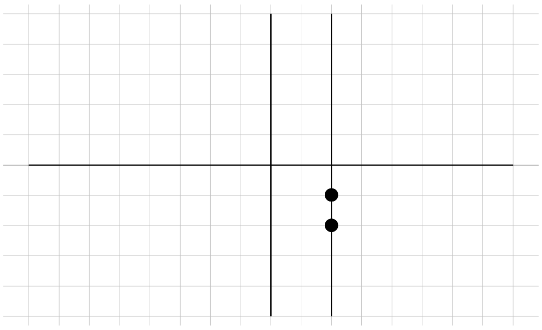

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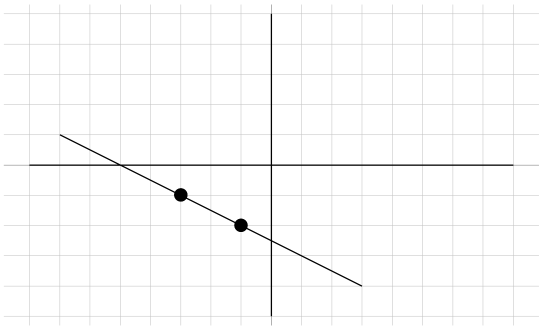

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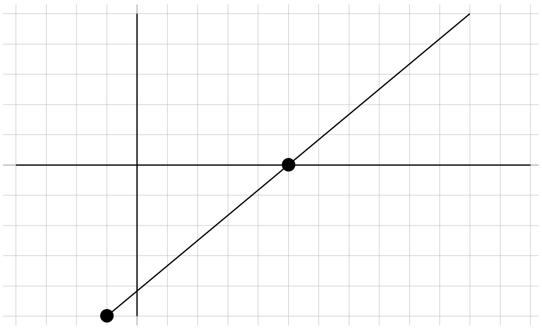

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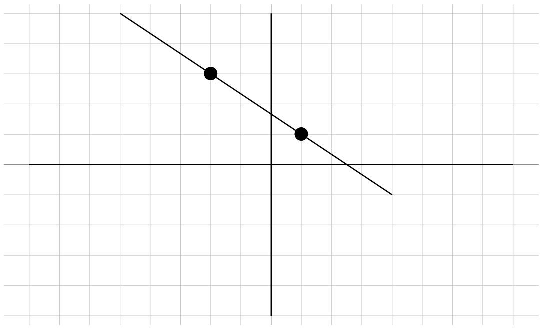

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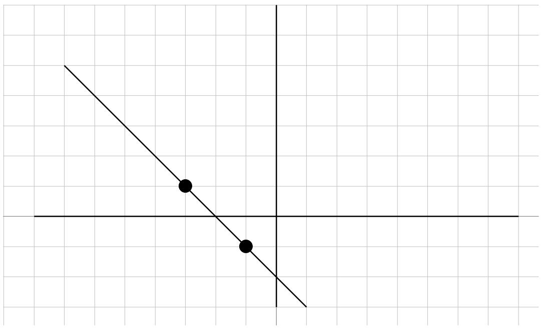

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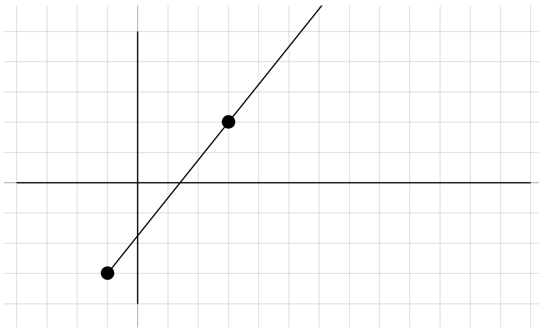

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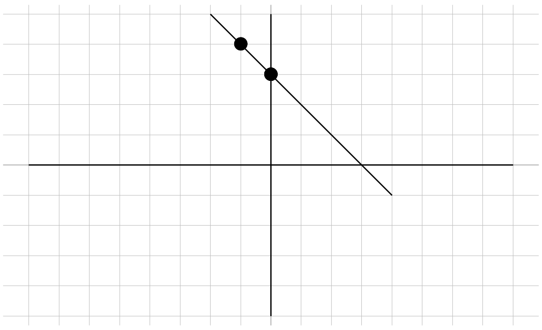

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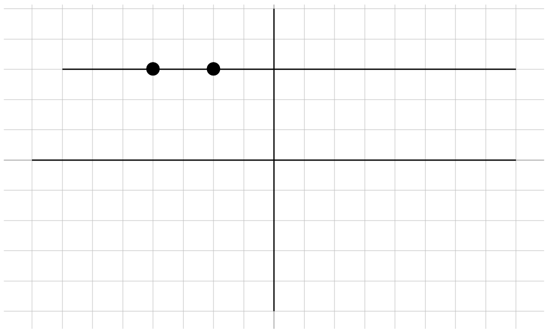


Find the slope of the line through each pair of points.

11) $(-2, 10), (-2, -15)$

12) $(1, 2), (-6, -14)$

13) $(-15, 10), (16, -7)$

14) $(13, -2), (7, 7)$

15) $(10, 18), (-11, -10)$

16) $(-3, 6), (-20, 13)$

17) $(-16, -14), (11, -14)$

18) $(13, 15), (2, 10)$

19) $(-4, 14), (-16, 8)$

20) $(9, -6), (-7, -7)$

21) $(12, -19), (6, 14)$

22) $(-16, 2), (15, -10)$

23) $(-5, -10), (-5, 20)$

24) $(8, 11), (-3, -13)$

25) $(-17, 19), (10, -7)$

26) $(11, -2), (1, 17)$

27) $(7, -14), (-8, -9)$

28) $(-18, -5), (14, -3)$

29) $(-5, 7), (-18, 14)$

30) $(19, 15), (5, 11)$

Find the value of x or y so that the line through the points has the given slope.

31) $(2, 6)$ and $(x, 2)$; slope: $\frac{4}{7}$

32) $(8, y)$ and $(-2, 4)$; slope: $-\frac{1}{5}$

33) $(-3, -2)$ and $(x, 6)$; slope: $-\frac{8}{5}$

34) $(-2, y)$ and $(2, 4)$; slope: $\frac{1}{4}$

35) $(-8, y)$ and $(-1, 1)$; slope: $\frac{6}{7}$

36) $(x, -1)$ and $(-4, 6)$; slope: $-\frac{7}{10}$

37) $(x, -7)$ and $(-9, -9)$; slope: $\frac{2}{5}$

38) $(2, -5)$ and $(3, y)$; slope: 6

39) $(x, 5)$ and $(8, 0)$; slope: $-\frac{5}{6}$

40) $(6, 2)$ and $(x, 6)$; slope: $-\frac{4}{5}$

0.3 Slope-Intercept Form

Objective: Give the equation of a line with a known slope and y-intercept.

When graphing a line we found one method we could use is to make a table of values. However, if we can identify some properties of the line, we may be able to make a graph much quicker and easier. One such method is finding the slope and the y-intercept of the equation. The slope can be represented by m and the y-intercept, where it crosses the axis and $x=0$, can be represented by $(0,b)$ where b is the value where the graph crosses the vertical y-axis. Any other point on the line can be represented by (x,y). Using this information we will look at the slope formula and solve the formula for y.

Example 132.

$$\begin{array}{rl} m,(0,b),(x,y) & \text{Using the slope formula gives:} \\ \dfrac{y-b}{x-0}=m & \text{Simplify} \\ \dfrac{y-b}{x}=m & \text{Multiply both sides by } x \\ y-b=mx & \text{Add } b \text{ to both sides} \\ \underline{+b \quad +b} & \\ y=mx+b & \text{Our Solution} \end{array}$$

This equation, $y=mx+b$ can be thought of as the equation of any line that as a slope of m and a y-intercept of b. This formula is known as the slope-intercept equation.

$$\textbf{Slope} - \textbf{Intercept Equation: } \boldsymbol{y=mx+b}$$

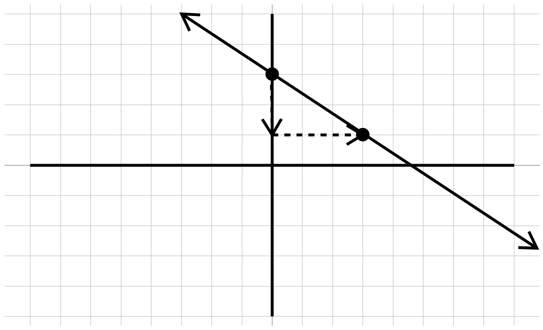
If we know the slope and the y-intercept we can easily find the equation that represents the line.

Example 133.

$$\begin{array}{rl} \text{Slope}=\dfrac{3}{4},\, y-\text{intercept}=-3 & \text{Use the slope} - \text{intercept equation} \\ y=mx+b & m \text{ is the slope, } b \text{ is the } y-\text{intercept} \\ y=\dfrac{3}{4}x-3 & \text{Our Solution} \end{array}$$

We can also find the equation by looking at a graph and finding the slope and y-intercept.

Example 134.



Identify the point where the graph crosses the y-axis (0,3). This means the y-intercept is 3.

Idenfity one other point and draw a slope triangle to find the slope. The slope is $-\frac{2}{3}$

$y = mx + b$ Slope-intercept equation

$$y = -\frac{2}{3}x + 3 \quad \text{Our Solution}$$

We can also move the opposite direction, using the equation identify the slope and y-intercept and graph the equation from this information. However, it will be important for the equation to first be in slope intercept form. If it is not, we will have to solve it for y so we can identify the slope and the y-intercept.

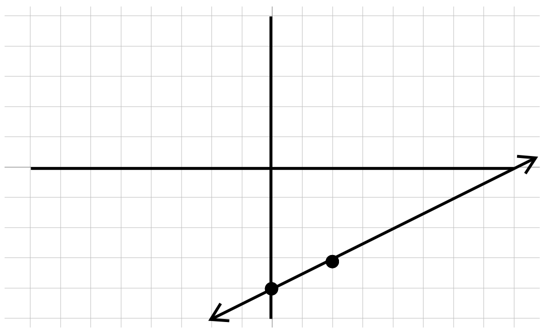
Example 135.

$$\begin{array}{rl}
\text{Write in slope} - \text{intercept form: } 2x - 4y = 6 & \text{Solve for } y \\
-2x \qquad -2x & \text{Subtract } 2x \text{ from both sides} \\
-4y = -2x + 6 & \text{Put } x \text{ term first} \\
\overline{-4} \qquad \overline{-4}\ \overline{-4} & \text{Divide each term by } -4 \\
y = \frac{1}{2}x - \frac{3}{2} & \text{Our Solution}
\end{array}$$

Once we have an equation in slope-intercept form we can graph it by first plotting the y-intercept, then using the slope, find a second point and connecting the dots.

Example 136.

$$\begin{array}{rl}
\text{Graph } y = \frac{1}{2}x - 4 & \text{Recall the slope} - \text{intercept formula} \\
y = mx + b & \text{Idenfity the slope, } m, \text{ and the } y - \text{intercept, } b \\
m = \frac{1}{2}, b = -4 & \text{Make the graph}
\end{array}$$



Starting with a point at the y-intercept of -4,

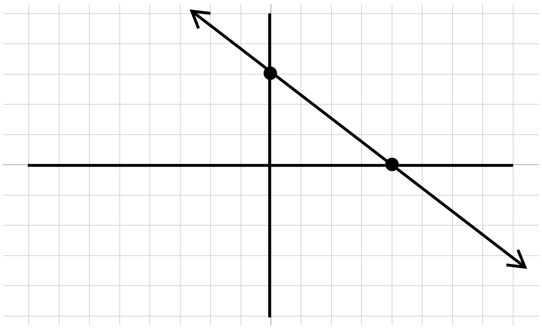
Then use the slope $\frac{\text{rise}}{\text{run}}$, so we will rise 1 unit and run 2 units to find the next point.

Once we have both points, connect the dots to get our graph.

World View Note: Before our current system of graphing, French Mathematician Nicole Oresme, in 1323 sugggested graphing lines that would look more like a bar graph with a constant slope!

Example 137.

$$\begin{array}{rl}
\text{Graph } 3x+4y=12 & \text{Not in slope intercept form} \\
-3x \qquad -3x & \text{Subtract } 3x \text{ from both sides} \\
4y=-3x+12 & \text{Put the } x \text{ term first} \\
\overline{4} \qquad \overline{4} \quad \overline{4} & \text{Divide each term by } 4 \\
y=-\dfrac{3}{4}x+3 & \text{Recall slope} - \text{intercept equation} \\
y=mx+b & \text{Idenfity } m \text{ and } b \\
m=-\dfrac{3}{4}, b=3 & \text{Make the graph}
\end{array}$$



Starting with a point at the y-intercept of 3,

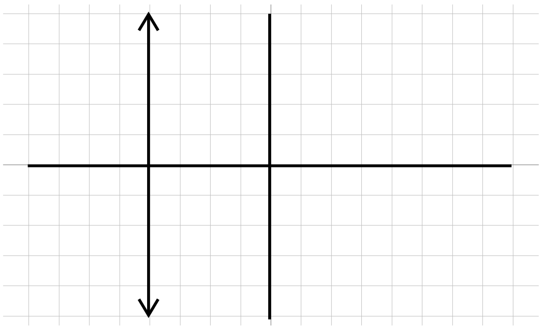
Then use the slope $\frac{\text{rise}}{\text{run}}$, but its negative so it will go downhill, so we will drop 3 units and run 4 units to find the next point.

Once we have both points, connect the dots to get our graph.

We want to be very careful not to confuse using slope to find the next point with use a coordinate such as $(4, -2)$ to find an individule point. Coordinates such as $(4, -2)$ start from the origin and move horizontally first, and vertically second. Slope starts from a point on the line that could be anywhere on the graph. The numerator is the vertical change and the denominator is the horizontal change.

Lines with zero slope or no slope can make a problem seem very different. Zero slope, or horiztonal line, will simply have a slope of zero which when multiplied by x gives zero. So the equation simply becomes $y=b$ or y is equal to the y-coordinate of the graph. If we have no slope, or a vertical line, the equation can't be written in slope intercept at all because the slope is undefined. There is no y in these equations. We will simply make x equal to the x-coordinate of the graph.

Example 138.



Give the equation of the line in the graph.

Because we have a vertical line and no slope there is no slope-intercept equation we can use. Rather we make x equal to the x-coordinate of -4

$x=-4 \qquad$ Our Solution

0.3 Practice - Slope-Intercept

Write the slope-intercept form of the equation of each line given the slope and the y-intercept.

1) Slope $= 2$, y-intercept $= 5$

2) Slope $= -6$, y-intercept $= 4$

3) Slope $= 1$, y-intercept $= -4$

4) Slope $= -1$, y-intercept $= -2$

5) Slope $= -\frac{3}{4}$, y-intercept $= -1$

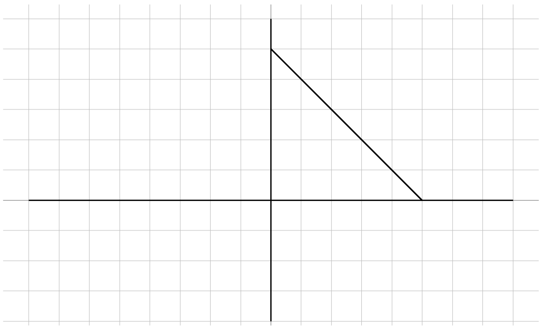
6) Slope $= -\frac{1}{4}$, y-intercept $= 3$

7) Slope $= \frac{1}{3}$, y-intercept $= 1$

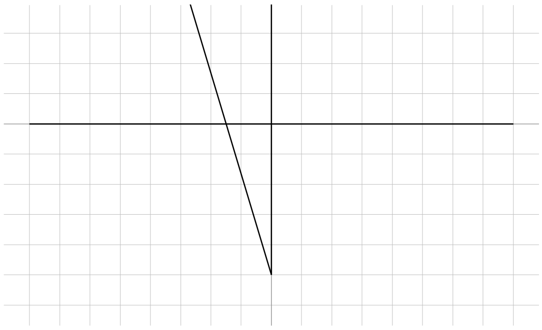
8) Slope $= \frac{2}{5}$, y-intercept $= 5$

Write the slope-intercept form of the equation of each line.

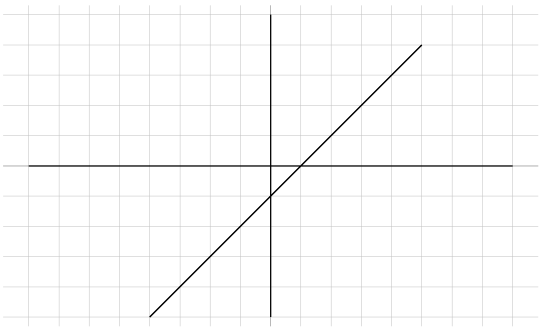
9)



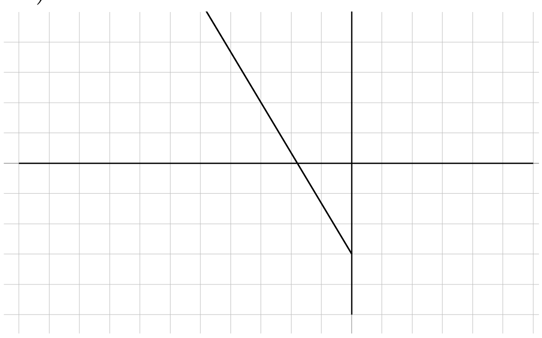
10)



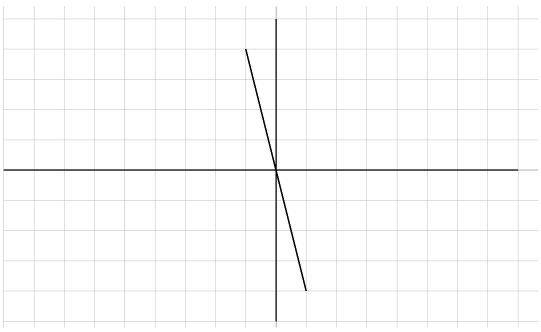
11)



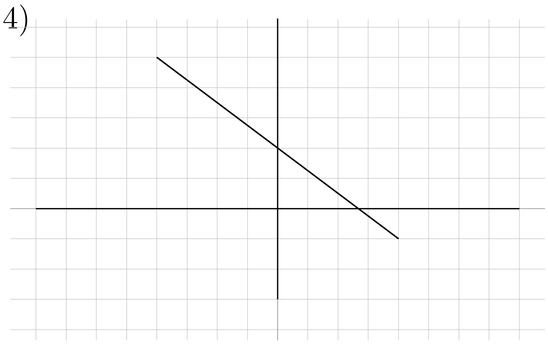
12)



13)



14)



15) $x + 10y = -37$

16) $x - 10y = 3$

17) $2x + y = -1$

18) $6x - 11y = -70$

19) $7x - 3y = 24$

20) $4x + 7y = 28$

21) $x = -8$

22) $x - 7y = -42$

23) $y - 4 = -(x + 5)$

24) $y - 5 = \frac{5}{2}(x - 2)$

25) $y - 4 = 4(x - 1)$

26) $y - 3 = -\frac{2}{3}(x + 3)$

27) $y + 5 = -4(x - 2)$

28) $0 = x - 4$

29) $y + 1 = -\frac{1}{2}(x - 4)$

30) $y + 2 = \frac{6}{5}(x + 5)$

Sketch the graph of each line.

31) $y = \frac{1}{3}x + 4$

32) $y = -\frac{1}{5}x - 4$

33) $y = \frac{6}{5}x - 5$

34) $y = -\frac{3}{2}x - 1$

35) $y = \frac{3}{2}x$

36) $y = -\frac{3}{4}x + 1$

37) $x - y + 3 = 0$

38) $4x + 5 = 5y$

39) $-y - 4 + 3x = 0$

40) $-8 = 6x - 2y$

41) $-3y = -5x + 9$

42) $-3y = 3 - \frac{3}{2}x$

0.4 Point-Slope Form

Objective: Give the equation of a line with a known slope and point.

The slope-intercept form has the advantage of being simple to remember and use, however, it has one major disadvantage: we must know the y-intercept in order to use it! Generally we do not know the y-intercept, we only know one or more points (that are not the y-intercept). In these cases we can't use the slope intercept equation, so we will use a different more flexible formula. If we let the slope of an equation be m, and a specific point on the line be (x_1, y_1), and any other point on the line be (x, y). We can use the slope formula to make a second equation.

Example 139.

$$\begin{array}{rl}
m, (x_1, y_1), (x, y) & \text{Recall slope formula} \\
\dfrac{y_2 - y_1}{x_2 - x_1} = m & \text{Plug in values} \\
\dfrac{y - y_1}{x - x_1} = m & \text{Multiply both sides by } (x - x_1) \\
y - y_1 = m(x - x_1) & \text{Our Solution}
\end{array}$$

If we know the slope, m of an equation and any point on the line (x_1, y_1) we can easily plug these values into the equation above which will be called the point-slope formula.

$$\textbf{Point} - \textbf{Slope Formula: } \boldsymbol{y - y_1 = m(x - x_1)}$$

Example 140.

Write the equation of the line through the point $(3, -4)$ with a slope of $\frac{3}{5}$.

$$\begin{array}{rl}
y - y_1 = m(x - x_1) & \text{Plug values into point} - \text{slope formula} \\
y - (-4) = \dfrac{3}{5}(x - 3) & \text{Simplify signs} \\
y + 4 = \dfrac{3}{5}(x - 3) & \text{Our Solution}
\end{array}$$

Often, we will prefer final answers be written in slope intercept form. If the directions ask for the answer in slope-intercept form we will simply distribute the slope, then solve for y.

Example 141.

Write the equation of the line through the point $(-6, 2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

$$\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Plug values into point} - \text{slope formula}\\
y - 2 &= -\frac{2}{3}(x - (-6)) && \text{Simplify signs}\\
y - 2 &= -\frac{2}{3}(x + 6) && \text{Distribute slope}\\
y - 2 &= -\frac{2}{3}x - 4 && \text{Solve for } y\\
\underline{+2} &\quad \underline{\quad +2} &&\\
y &= -\frac{2}{3}x - 2 && \text{Our Solution}
\end{aligned}$$

An important thing to observe about the point slope formula is that the operation between the x's and y's is subtraction. This means when you simplify the signs you will have the opposite of the numbers in the point. We need to be very careful with signs as we use the point-slope formula.

In order to find the equation of a line we will always need to know the slope. If we don't know the slope to begin with we will have to do some work to find it first before we can get an equation.

Example 142.

Find the equation of the line through the points $(-2, 5)$ and $(4, -3)$.

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{First we must find the slope}\\
m = \frac{-3 - 5}{4 - (-2)} &= \frac{-8}{6} = -\frac{4}{3} && \text{Plug values in slope formula and evaluate}\\
y - y_1 &= m(x - x_1) && \text{With slope and either point, use point} - \text{slope formula}\\
y - 5 &= -\frac{4}{3}(x - (-2)) && \text{Simplify signs}\\
y - 5 &= -\frac{4}{3}(x + 2) && \text{Our Solution}
\end{aligned}$$

Example 143.

Find the equation of the line through the points $(-3,4)$ and $(-1,-2)$ in slope-intercept form.

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{First we must find the slope}\\
m = \frac{-2-4}{-1-(-3)} &= \frac{-6}{2} = -3 && \text{Plug values in slope formula and evaluate}\\
y - y_1 &= m(x - x_1) && \text{With slope and either point, point} - \text{slope formula}\\
y - 4 &= -3(x - (-3)) && \text{Simplify signs}\\
y - 4 &= -3(x + 3) && \text{Distribute slope}\\
y - 4 &= -3x - 9 && \text{Solve for } y\\
+4 \quad & \quad\;\; +4 && \text{Add 4 to both sides}\\
y &= -3x - 5 && \text{Our Solution}
\end{aligned}$$

Example 144.

Find the equation of the line through the points $(6,-2)$ and $(-4,1)$ in slope-intercept form.

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{First we must find the slope}\\
m = \frac{1-(-2)}{-4-6} &= \frac{3}{-10} = -\frac{3}{10} && \text{Plug values into slope formula and evaluate}\\
y - y_1 &= m(x - x_1) && \text{Use slope and either point, use point} - \text{slope formula}\\
y - (-2) &= -\frac{3}{10}(x - 6) && \text{Simplify signs}\\
y + 2 &= -\frac{3}{10}(x - 6) && \text{Distribute slope}\\
y + 2 &= -\frac{3}{10}x + \frac{9}{5} && \text{Solve for } y\text{. Subtract 2 from both sides}\\
-2 \quad & \qquad\quad -\frac{10}{5} && \text{Using } \frac{10}{5} \text{ on right so we have } a \text{ common denominator}\\
y &= -\frac{3}{10}x - \frac{1}{5} && \text{Our Solution}
\end{aligned}$$

World View Note: The city of Konigsberg (now Kaliningrad, Russia) had a river that flowed through the city breaking it into several parts. There were 7 bridges that connected the parts of the city. In 1735 Leonhard Euler considered the question of whether it was possible to cross each bridge exactly once and only once. It turned out that this problem was impossible, but the work laid the foundation of what would become graph theory.

0.4 Practice - Point-Slope Form

Write the point-slope form of the equation of the line through the given point with the given slope.

1) through $(2,3)$, slope = undefined

2) through $(1,2)$, slope = undefined

3) through $(2,2)$, slope $=\frac{1}{2}$

4) through $(2,1)$, slope $=-\frac{1}{2}$

5) through $(-1,-5)$, slope $=9$

6) through $(2,-2)$, slope $=-2$

7) through $(-4,1)$, slope $=\frac{3}{4}$

8) through $(4,-3)$, slope $=-2$

9) through $(0,-2)$, slope $=-3$

10) through $(-1,1)$, slope $=4$

11) through $(0,-5)$, slope $=-\frac{1}{4}$

12) through $(0,2)$, slope $=-\frac{5}{4}$

13) through $(-5,-3)$, slope $=\frac{1}{5}$

14) through $(-1,-4)$, slope $=-\frac{2}{3}$

15) through $(-1,4)$, slope $=-\frac{5}{4}$

16) through $(1,-4)$, slope $=-\frac{3}{2}$

Write the slope-intercept form of the equation of the line through the given point with the given slope.

17) through: $(-1,-5)$, slope $=2$

18) through: $(2,-2)$, slope $=-2$

19) through: $(5,-1)$, slope $=-\frac{3}{5}$

20) through: $(-2,-2)$, slope $=-\frac{2}{3}$

21) through: $(-4,1)$, slope $=\frac{1}{2}$

22) through: $(4,-3)$, slope $=-\frac{7}{4}$

23) through: $(4,-2)$, slope $=-\frac{3}{2}$

24) through: $(-2,0)$, slope $=-\frac{5}{2}$

25) through: $(-5,-3)$, slope $=-\frac{2}{5}$

26) through: $(3,3)$, slope $=\frac{7}{3}$

27) through: $(2,-2)$, slope $=1$

28) through: $(-4,-3)$, slope $=0$

29) through: $(-3,4)$, slope = undefined

30) through: $(-2,-5)$, slope $=2$

31) through: $(-4,2)$, slope $=-\frac{1}{2}$

32) through: $(5,3)$, slope $=\frac{6}{5}$

Write the point-slope form of the equation of the line through the given points.

33) through: $(-4,3)$ and $(-3,1)$

34) through: $(1,3)$ and $(-3,3)$

35) through: $(5,1)$ and $(-3,0)$

36) through: $(-4,5)$ and $(4,4)$

37) through: $(-4,-2)$ and $(0,4)$

38) through: $(-4,1)$ and $(4,4)$

39) through: $(3,5)$ and $(-5,3)$

40) through: $(-1,-4)$ and $(-5,0)$

41) through: $(3,-3)$ and $(-4,5)$

42) through: $(-1,-5)$ and $(-5,-4)$

Write the slope-intercept form of the equation of the line through the given points.

43) through: $(-5,1)$ and $(-1,-2)$

44) through: $(-5,-1)$ and $(5,-2)$

45) through: $(-5,5)$ and $(2,-3)$

46) through: $(1,-1)$ and $(-5,-4)$

47) through: $(4,1)$ and $(1,4)$

48) through: $(0,1)$ and $(-3,0)$

49) through: $(0,2)$ and $(5,-3)$

50) through: $(0,2)$ and $(2,4)$

51) through: $(0,3)$ and $(-1,-1)$

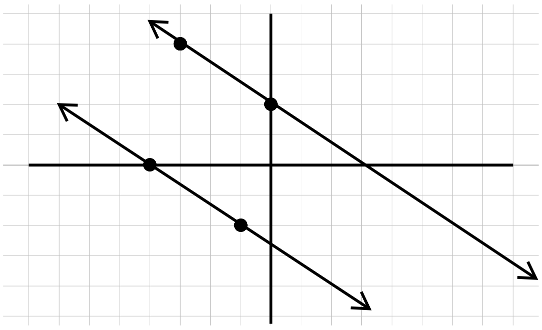
52) through: $(-2,0)$ and $(5,3)$

0.5 Parallel and Perpendicular Lines

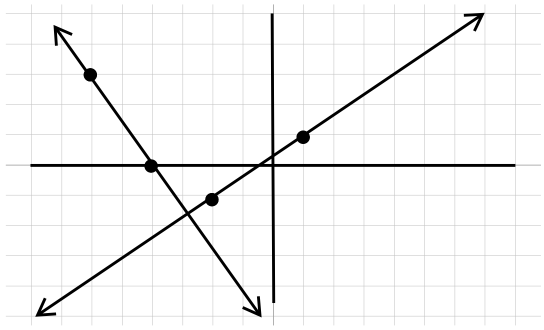
Objective: Identify the equation of a line given a parallel or perpendicular line.

There is an interesting connection between the slope of lines that are parallel and the slope of lines that are perpendicular (meet at a right angle). This is shown in the following example.

Example 145.



The above graph has two parallel lines. The slope of the top line is down 2, run 3, or $-\frac{2}{3}$. The slope of the bottom line is down 2, run 3 as well, or $-\frac{2}{3}$.



The above graph has two perpendicular lines. The slope of the flatter line is up 2, run 3 or $\frac{2}{3}$. The slope of the steeper line is down 3, run 2 or $-\frac{3}{2}$.

World View Note: Greek Mathematician Euclid lived around 300 BC and published a book titled, *The Elements*. In it is the famous parallel postulate which mathematicians have tried for years to drop from the list of postulates. The attempts have failed, yet all the work done has developed new types of geometries!

As the above graphs illustrate, parallel lines have the same slope and perpendicular lines have opposite (one positive, one negative) reciprocal (flipped fraction) slopes. We can use these properties to make conclusions about parallel and perpendicular lines.

Example 146.

Find the slope of a line parallel to $5y - 2x = 7$.

$$
\begin{array}{rl}
5y - 2x = 7 & \text{To find the slope we will put equation in slope} - \text{intercept form} \\
\underline{+2x + 2x} & \text{Add } 2x \text{ to both sides} \\
5y = 2x + 7 & \text{Put } x \text{ term first} \\
\overline{5} \quad \overline{5} \quad \overline{5} & \text{Divide each term by } 5 \\
y = \dfrac{2}{5}x + \dfrac{7}{5} & \text{The slope is the coefficient of } x
\end{array}
$$

$$m=\frac{2}{5}\quad\text{Slope of first line. Parallel lines have the same slope}$$

$$m=\frac{2}{5}\quad\text{Our Solution}$$

Example 147.

Find the slope of a line perpendicular to $3x-4y=2$

$$\begin{array}{rl}
3x-4y=2 & \text{To find slope we will put equation in slope} - \text{intercept form}\\
\underline{-3x\qquad -3x} & \text{Subtract } 3x \text{ from both sides}\\
-4y=-3x+2 & \text{Put } x \text{ term first}\\
\overline{-4}\quad\ \overline{-4}\ \overline{-4} & \text{Divide each term by } -4\\
y=\frac{3}{4}x-\frac{1}{2} & \text{The slope is the coefficient of } x\\
 & \\
m=\frac{3}{4} & \text{Slope of first lines. Perpendicular lines have opposite reciprocal slopes}\\
 & \\
m=-\frac{4}{3} & \text{Our Solution}
\end{array}$$

Once we have a slope, it is possible to find the complete equation of the second line if we know one point on the second line.

Example 148.

Find the equation of a line through $(4,-5)$ and parallel to $2x-3y=6$.

$$\begin{array}{rl}
2x-3y=6 & \text{We first need slope of parallel line}\\
\underline{-2x\qquad -2x} & \text{Subtract } 2x \text{ from each side}\\
-3y=-2x+6 & \text{Put } x \text{ term first}\\
\overline{-3}\quad\ \overline{-3}\,\overline{-3} & \text{Divide each term by } -3\\
y=\frac{2}{3}x-2 & \text{Identify the slope, the coefficient of } x\\
 & \\
m=\frac{2}{3} & \text{Parallel lines have the same slope}\\
 & \\
m=\frac{2}{3} & \text{We will use this slope and our point } (4,-5)\\
 & \\
y-y_1=m(x-x_1) & \text{Plug this information into point slope formula}\\
y-(-5)=\frac{2}{3}(x-4) & \text{Simplify signs}\\
 & \\
y+5=\frac{2}{3}(x-4) & \text{Our Solution}
\end{array}$$

Example 149.

Find the equation of the line through $(6,-9)$ perpendicular to $y=-\frac{3}{5}x+4$ in slope-intercept form.

$$\begin{aligned}
y &= -\frac{3}{5}x+4 && \text{Identify the slope, coefficient of } x\\
m &= -\frac{3}{5} && \text{Perpendicular lines have opposite reciprocal slopes}\\
m &= \frac{5}{3} && \text{We will use this slope and our point } (6,-9)\\
y-y_1 &= m(x-x_1) && \text{Plug this information into point} - \text{slope formula}\\
y-(-9) &= \frac{5}{3}(x-6) && \text{Simplify signs}\\
y+9 &= \frac{5}{3}(x-6) && \text{Distribute slope}\\
y+9 &= \frac{5}{3}x-10 && \text{Solve for } y\\
\underline{-9} &\quad \underline{\quad -9} && \text{Subtract 9 from both sides}\\
y &= \frac{5}{3}x-19 && \text{Our Solution}
\end{aligned}$$

Zero slopes and no slopes may seem like opposites (one is a horizontal line, one is a vertical line). Because a horizontal line is perpendicular to a vertical line we can say that no slope and zero slope are actually perpendicular slopes!

Example 150.

Find the equation of the line through $(3,4)$ perpendicular to $x=-2$

$$\begin{aligned}
x &= -2 && \text{This equation has no slope, } a \text{ vertical line}\\
&\text{no slope} && \text{Perpendicular line then would have } a \text{ zero slope}\\
m &= 0 && \text{Use this and our point } (3,4)\\
y-y_1 &= m(x-x_1) && \text{Plug this information into point} - \text{slope formula}\\
y-4 &= 0(x-3) && \text{Distribute slope}\\
y-4 &= 0 && \text{Solve for } y\\
&\underline{+4+4} && \text{Add 4 to each side}\\
y &= 4 && \text{Our Solution}
\end{aligned}$$

Being aware that to be perpendicular to a vertical line means we have a horizontal line through a y value of 4, thus we could have jumped from this point right to the solution, $y=4$.

0.5 Practice - Parallel and Perpendicular Lines

Find the slope of a line parallel to each given line.

1) $y = 2x + 4$

2) $y = -\frac{2}{3}x + 5$

3) $y = 4x - 5$

4) $y = -\frac{10}{3}x - 5$

5) $x - y = 4$

6) $6x - 5y = 20$

7) $7x + y = -2$

8) $3x + 4y = -8$

Find the slope of a line perpendicular to each given line.

9) $x = 3$

10) $y = -\frac{1}{2}x - 1$

11) $y = -\frac{1}{3}x$

12) $y = \frac{4}{5}x$

13) $x - 3y = -6$

14) $3x - y = -3$

15) $x + 2y = 8$

16) $8x - 3y = -9$

Write the point-slope form of the equation of the line described.

17) through: $(2, 5)$, parallel to $x = 0$

18) through: $(5, 2)$, parallel to $y = \frac{7}{5}x + 4$

19) through: $(3, 4)$, parallel to $y = \frac{9}{2}x - 5$

20) through: $(1, -1)$, parallel to $y = -\frac{3}{4}x + 3$

21) through: $(2, 3)$, parallel to $y = \frac{7}{5}x + 4$

22) through: $(-1, 3)$, parallel to $y = -3x - 1$

23) through: $(4, 2)$, parallel to $x = 0$

24) through: $(1, 4)$, parallel to $y = \frac{7}{5}x + 2$

25) through: $(1, -5)$, perpendicular to $-x + y = 1$

26) through: $(1, -2)$, perpendicular to $-x + 2y = 2$

27) through: $(5, 2)$, perpendicular to $5x + y = -3$

28) through: $(1, 3)$, perpendicular to $-x + y = 1$

29) through: $(4, 2)$, perpendicular to $-4x + y = 0$

30) through: $(-3, -5)$, perpendicular to $3x + 7y = 0$

31) through: $(2, -2)$ perpendicular to $3y - x = 0$

32) through: $(-2, 5)$. perpendicular to $y - 2x = 0$

Write the slope-intercept form of the equation of the line described.

33) through: $(4, -3)$, parallel to $y = -2x$

34) through: $(-5, 2)$, parallel to $y = \frac{3}{5}x$

35) through: $(-3, 1)$, parallel to $y = -\frac{4}{3}x - 1$

36) through: $(-4, 0)$, parallel to $y = -\frac{5}{4}x + 4$

37) through: $(-4, -1)$, parallel to $y = -\frac{1}{2}x + 1$

38) through: $(2, 3)$, parallel to $y = \frac{5}{2}x - 1$

39) through: $(-2, -1)$, parallel to $y = -\frac{1}{2}x - 2$

40) through: $(-5, -4)$, parallel to $y = \frac{3}{5}x - 2$

41) through: $(4, 3)$, perpendicular to $x + y = -1$

42) through: $(-3, -5)$, perpendicular to $x + 2y = -4$

43) through: $(5, 2)$, perpendicular to $x = 0$

44) through: $(5, -1)$, perpendicular to $-5x + 2y = 10$

45) through: $(-2, 5)$, perpendicular to $-x + y = -2$

46) through: $(2, -3)$, perpendicular to $-2x + 5y = -10$

47) through: $(4, -3)$, perpendicular to $-x + 2y = -6$

48) through: $(-4, 1)$, perpendicular to $4x + 3y = -9$

Chapter 1 - Systems of Equations

1.1 Graphing

Objective: Solve systems of equations by graphing and identifying the point of intersection.

We have solved problems like $3x - 4 = 11$ by adding 4 to both sides and then dividing by 3 (solution is $x = 5$). We also have methods to solve equations with more than one variable in them. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables such as x and y we will need two equations. When we have several equations we are using to solve, we call the equations a **system of equations**. When solving a system of equations we are looking for a solution that works in both equations. This solution is usually given as an ordered pair (x, y). The following example illustrates a solution working in both equations

Example 1.

Show (2,1) is the solution to the system $\begin{array}{l} 3x - y = 5 \\ x + y = 3 \end{array}$

$$\begin{array}{rl}
(2,1) & \text{Identify } x \text{ and } y \text{ from the orderd pair} \\
x = 2, y = 1 & \text{Plug these values into each equation} \\
& \\
3(2) - (1) = 5 & \text{First equation} \\
6 - 1 = 5 & \text{Evaluate} \\
5 = 5 & \text{True} \\
& \\
(2) + (1) = 3 & \text{Second equation, evaluate} \\
3 = 3 & \text{True}
\end{array}$$

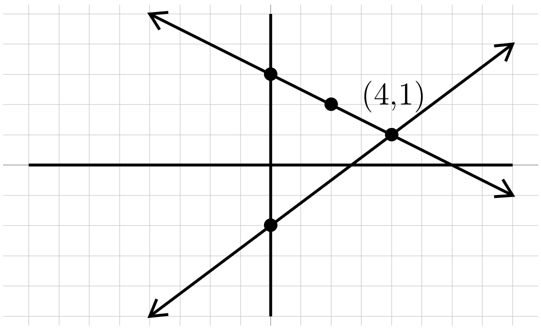
As we found a true statement for both equations we know (2,1) is the solution to the system. It is in fact the only combination of numbers that works in both equations. In this lesson we will be working to find this point given the equations. It seems to follow that if we use points to describe the solution, we can use graphs to find the solutions.

If the graph of a line is a picture of all the solutions, we can graph two lines on the same coordinate plane to see the solutions of both equations. We are interested in the point that is a solution for both lines, this would be where the lines intersect! If we can find the intersection of the lines we have found the solution that works in both equations.

Example 2.

$$\begin{aligned} y &= -\frac{1}{2}x + 3 \\ y &= \frac{3}{4}x - 2 \end{aligned} \qquad \text{To graph we identify slopes and } y - \text{intercepts}$$

$$\begin{aligned} \text{First: } m &= -\frac{1}{2}, b = 3 \\ \text{Second: } m &= \frac{3}{4}, b = -2 \end{aligned} \qquad \text{Now we can graph both lines on the same plane.}$$



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Remember a negative slope is downhill!

Find the intersection point, (4,1)

(4,1) Our Solution

Often our equations won't be in slope-intercept form and we will have to solve both equations for y first so we can idenfity the slope and y-intercept.

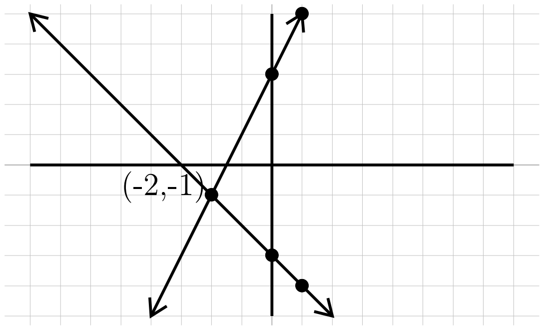
Example 3.

$$\begin{aligned} 6x - 3y &= -9 \\ 2x + 2y &= -6 \end{aligned} \qquad \text{Solve each equation for } y$$

$$\begin{array}{cc|l}
6x - 3y = -9 & 2x + 2y = -6 & \\
\underline{-6x \qquad -6x} & \underline{-2x \qquad -2x} & \text{Subtract } x \text{ terms} \\
-3y = -6x - 9 & 2y = -2x - 6 & \text{Put } x \text{ terms first} \\
\overline{-3} \quad \overline{-3} \ \overline{-3} & \overline{2} \quad \overline{2} \ \overline{2} & \text{Divide by coefficient of } y \\
y = 2x + 3 & y = -x - 3 & \text{Identify slope and } y - \text{intercepts}
\end{array}$$

$$\text{First: } m=\frac{2}{1}, b=3 \qquad \text{Second: } m=-\frac{1}{1}, b=-3$$

Now we can graph both lines on the same plane



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Remember a negative slope is downhill!

Find the intersection point, $(-2,-1)$

$(-2,-1)$ Our Solution

As we are graphing our lines, it is possible to have one of two unexpected results. These are shown and discussed in the next two example.

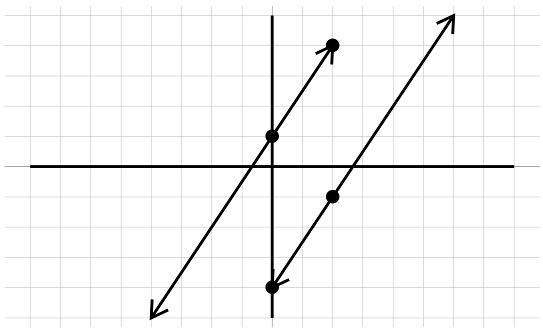
Example 4.

$$y=\frac{3}{2}x-4$$
$$y=\frac{3}{2}x+1$$

Identify slope and $y-$intercept of each equation

$$\text{First: } m=\frac{3}{2}, b=-4 \qquad \text{Second: } m=\frac{3}{2}, b=1$$

Now we can graph both equations on the same plane



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

The two lines do not intersect! They are parallel! If the lines do not intersect we know that there is no point that works in both equations, there is no solution

$\varnothing$ No Solution

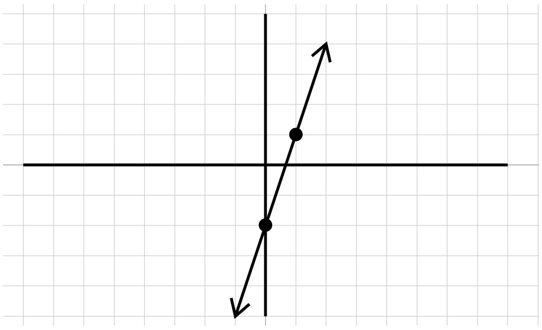
We also could have noticed that both lines had the same slope. Remembering that parallel lines have the same slope we would have known there was no solution even without having to graph the lines.

Example 5.

$$\begin{array}{r} 2x-6y=12 \\ 3x-9y=18 \end{array} \quad \text{Solve each equation for } y$$

$$\begin{array}{rrl}
2x-6y=12 & 3x-9y=18 & \\
\underline{-2x \qquad -2x} & \underline{-3x \qquad -3x} & \text{Subtract } x \text{ terms} \\
-6y=-2x+12 & -9y=-3x+18 & \text{Put } x \text{ terms first} \\
\overline{-6} \quad \overline{-6} \quad \overline{-6} & \overline{-9} \quad \overline{-9} \quad \overline{-9} & \text{Divide by coefficient of } y \\
y=\frac{1}{3}x-2 & y=\frac{1}{3}x-2 & \text{Identify the slopes and } y-\text{intercepts} \\
\end{array}$$

$$\begin{array}{rl}
\text{First: } m=\frac{1}{3}, b=-2 & \\
& \text{Now we can graph both equations together} \\
\text{Second: } m=\frac{1}{3}, b=-2 &
\end{array}$$



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Both equations are the same line! As one line is directly on top of the other line, we can say that the lines "intersect" at all the points! Here we say we have infinite solutions

Once we had both equations in slope-intercept form we could have noticed that the equations were the same. At this point we could have stated that there are infinite solutions without having to go through the work of graphing the equations.

World View Note: The Babylonians were the first to work with systems of equations with two variables. However, their work with systems was quickly passed by the Greeks who would solve systems of equations with three or four variables and around 300 AD, developed methods for solving systems with any number of unknowns!

1.1 Practice - Graphing

Solve each equation by graphing.

1) $y=-x+1$
$y=-5x-3$

2) $y=-\frac{5}{4}x-2$
$y=-\frac{1}{4}x+2$

3) $y=-3$
$y=-x-4$

4) $y=-x-2$
$y=\frac{2}{3}x+3$

5) $y=-\frac{3}{4}x+1$
$y=-\frac{3}{4}x+2$

6) $y=2x+2$
$y=-x-4$

7) $y=\frac{1}{3}x+2$
$y=-\frac{5}{3}x-4$

8) $y=2x-4$
$y=\frac{1}{2}x+2$

9) $y=\frac{5}{3}x+4$
$y=-\frac{2}{3}x-3$

10) $y=\frac{1}{2}x+4$
$y=\frac{1}{2}x+1$

11) $x+3y=-9$
$5x+3y=3$

12) $x+4y=-12$
$2x+y=4$

13) $x-y=4$
$2x+y=-1$

14) $6x+y=-3$
$x+y=2$

15) $2x+3y=-6$
$2x+y=2$

16) $3x+2y=2$
$3x+2y=-6$

17) $2x+y=2$
$x-y=4$

18) $x+2y=6$
$5x-4y=16$

19) $2x+y=-2$
$x+3y=9$

20) $x-y=3$
$5x+2y=8$

21) $0=-6x-9y+36$
$12=6x-3y$

22) $-2y+x=4$
$2=-x+\frac{1}{2}y$

23) $2x-y=-1$
$0=-2x-y-3$

24) $-2y=-4-x$
$-2y=-5x+4$

25) $3+y=-x$
$-4-6x=-y$

26) $16=-x-4y$
$-2x=-4-4y$

27) $-y+7x=4$
$-y-3+7x=0$

28) $-4+y=x$
$x+2=-y$

29) $-12+x=4y$
$12-5x=4y$

30) $-5x+1=-y$
$-y+x=-3$

4.1

Answers - Graphing

1) $(-1,2)$
2) $(-4,3)$
3) $(-1,-3)$
4) $(-3,1)$
5) No Solution
6) $(-2,-2)$
7) $(-3,1)$
8) $(4,4)$
9) $(-3,-1)$
10) No Solution
11) $(3,-4)$
12) $(4,-4)$
13) $(1,-3)$
14) $(-1,3)$
15) $(3,-4)$
16) No Solution
17) $(2,-2)$
18) $(4,1)$
19) $(-3,4)$
20) $(2,-1)$
21) $(3,2)$
22) $(-4,-4)$
23) $(-1,-1)$
24) $(2,3)$
25) $(-1,-2)$
26) $(-4,-3)$
27) No Solution
28) $(-3,1)$
29) $(4,-2)$
30) $(1,4)$

1.2 Substitution

Objective: Solve systems of equations using substitution.

When solving a system by graphing has several limitations. First, it requires the graph to be perfectly drawn, if the lines are not straight we may arrive at the wrong answer. Second, graphing is not a great method to use if the answer is really large, over 100 for example, or if the answer is a decimal the that graph will not help us find, 3.2134 for example. For these reasons we will rarely use graphing to solve our systems. Instead, an algebraic approach will be used.

The first algebraic approach is called substitution. We will build the concepts of substitution through several example, then end with a five-step process to solve problems using this method.

Example 1.

$x=5$ $y=2x-3$	We already know $x=5$, substitute this into the other equation
$y=2(\mathbf{5})-3$	Evaluate, multiply first
$y=10-3$	Subtract
$y=7$	We now also have y
$(5,7)$	Our Solution

When we know what one variable equals we can plug that value (or expression) in for the variable in the other equation. It is very important that when we substitute, the substituted value goes in parenthesis. The reason for this is shown in the next example.

Example 2.

$2x-3y=7$ $y=3x-7$	We know $y=3x-7$, substitute this into the other equation
$2x-3(\boldsymbol{3x-7})=7$	Solve this equation, distributing -3 first

$$
\begin{array}{rl}
2x-9x+21=7 & \text{Combine like terms } 2x-9x \\
-7x+21=7 & \text{Subtract } 21 \\
\underline{\mathbf{-21\ \ -21}} & \\
-7x=-14 & \text{Divide by } -7 \\
\overline{\mathbf{-7}}\quad\ \overline{\mathbf{-7}} & \\
x=2 & \text{We now have our } x\text{, plug into the } y= \text{equation to find } y \\
y=3(\mathbf{2})-7 & \text{Evaluate, multiply first} \\
y=6-7 & \text{Subtract} \\
y=-1 & \text{We now also have } y \\
(2,-1) & \text{Our Solution}
\end{array}
$$

By using the entire expression $3x-7$ to replace y in the other equation we were able to reduce the system to a single linear equation which we can easily solve for our first variable. However, the lone variable (a variable without a coefficient) is not always alone on one side of the equation. If this happens we can isolate it by solving for the lone variable.

Example 3.

$$
\begin{array}{rl}
3x+2y=1 & \\
\boldsymbol{x}-5y=6 & \text{Lone variable is } x\text{, isolate by adding } 5y \text{ to both sides.} \\
\underline{\mathbf{+5y+5y}} & \\
x=6+5y & \text{Substitute this into the untouched equation} \\
3(\mathbf{6+5y})+2y=1 & \text{Solve this equation, distributing 3 first} \\
18+15y+2y=1 & \text{Combine like terms } 15y+2y \\
18+17y=1 & \text{Subtract 18 from both sides} \\
\underline{\mathbf{-18\qquad -18}} & \\
17y=-17 & \text{Divide both sides by 17} \\
\overline{\mathbf{17}}\quad\ \overline{\mathbf{17}} & \\
y=-1 & \text{We have our } y\text{, plug this into the } x= \text{equation to find } x \\
x=6+5(\mathbf{-1}) & \text{Evaluate, multiply first} \\
x=6-5 & \text{Subtract} \\
x=1 & \text{We now also have } x \\
(1,-1) & \text{Our Solution}
\end{array}
$$

The process in the previous example is how we will solve problems using substitution. This process is described and illustrated in the following table which lists the five steps to solving by substitution.

Problem	$4x-2y=2$ $2x+y=-5$
1. Find the lone variable	Second Equation, y $2x+\boldsymbol{y}=-5$
2. Solve for the lone variable	$-2x \qquad -2x$ $\boldsymbol{y=-5-2x}$
3. Substitute into the untouched equation	$4x-2(\boldsymbol{-5-2x})=2$
4. Solve	$4x+10+4x=2$ $8x+10=2$ $\underline{\quad -10 \ -10}$ $\frac{8x}{8}=\frac{-8}{8}$ $\boldsymbol{x=-1}$
5. Plug into lone variable equation and evaluate	$y=-5-2(\boldsymbol{-1})$ $y=-5+2$ $\boldsymbol{y=-3}$
Solution	$(-1,-3)$

Sometimes we have several lone variables in a problem. In this case we will have the choice on which lone variable we wish to solve for, either will give the same final result.

Example 4.

$$\begin{array}{rl}
\boldsymbol{x}+y=5 & \text{Find the lone variable: } x \text{ or } y \text{ in first, or } x \text{ in second.} \\
x-y=-1 & \text{We will chose } x \text{ in the first} \\
\boldsymbol{x}+y=5 & \text{Solve for the lone variable, subtract } y \text{ from both sides} \\
\underline{\boldsymbol{-y \ -y}} & \\
x=5-y & \text{Plug into the untouched equation, the second equation} \\
(\boldsymbol{5-y})-y=-1 & \text{Solve, parenthesis are not needed here, combine like terms} \\
5-2y=-1 & \text{Subtract 5 from both sides} \\
\underline{-5 \qquad -5} & \\
-2y=-6 & \text{Divide both sides by } -2 \\
\overline{-2} \quad \overline{-2} & \\
y=3 & \text{We have our } y! \\
x=5-(3) & \text{Plug into lone variable equation, evaluate} \\
x=2 & \text{Now we have our } x
\end{array}$$

$(2,3)$ Our Solution

Just as with graphing it is possible to have no solution $\varnothing$ (parallel lines) or infinite solutions (same line) with the substitution method. While we won't have a parallel line or the same line to look at and conclude if it is one or the other, the process takes an interesting turn as shown in the following example.

Example 5.

$\boldsymbol{y}+4=3x$ $2y-6x=-8$	Find the lone variable, y in the first equation
$y+4=3x$	Solve for the lone variable, subtract 4 from both sides
$\underline{-4\quad -4}$	
$y=3x-4$	Plug into untouched equation
$2(\mathbf{3x-4})-6x=-8$	Solve, distribute through parenthesis
$6x-8-6x=-8$	Combine like terms $6x-6x$
$-8=-8$	Variables are gone! A true statement.
Infinite solutions	Our Solution

Because we had a true statement, and no variables, we know that anything that works in the first equation, will also work in the second equation. However, we do not always end up with a true statement.

Example 6.

$6x-3y=-9$ $-2x+\boldsymbol{y}=5$	Find the lone variable, y in the second equation
$-2x+y=5$	Solve for the lone variable, add $2x$ to both sides
$\underline{+2x\quad +2x}$	
$y=5+2x$	Plug into untouched equation
$6x-3(\mathbf{5+2x})=-9$	Solve, distribute through parenthesis
$6x-15-6x=-9$	Combine like terms $6x-6x$
$-15\neq-9$	Variables are gone! A false statement.
No Solution $\varnothing$	Our Solution

Because we had a false statement, and no variables, we know that nothing will work in both equations.

World View Note: French mathematician Rene Descartes wrote a book which included an appendix on geometry. It was in this book that he suggested using letters from the end of the alphabet for unknown values. This is why often we are solving for the variables $x, y,$ and z.

One more question needs to be considered, what if there is no lone variable? If there is no lone variable substitution can still work to solve, we will just have to select one variable to solve for and use fractions as we solve.

Example 7.

$5\boldsymbol{x} - 6y = -14$	No lone variable,
$-2x + 4y = 12$	we will solve for x in the first equation
$5x - 6y = -14$	Solve for our variable, add $6y$ to both sides
$\underline{\ \boldsymbol{+6y}\ \ \boldsymbol{+6y}}$	
$\dfrac{5x}{\mathbf{5}} = \dfrac{-14}{\mathbf{5}} + \dfrac{6y}{\mathbf{5}}$	Divide each term by 5
$x = \dfrac{-14}{5} + \dfrac{6y}{5}$	Plug into untouched equation
$-2\left(\dfrac{-14}{5} + \dfrac{6y}{5}\right) + 4y = 12$	Solve, distribute through parenthesis
$\dfrac{28}{5} - \dfrac{12y}{5} + 4y = 12$	Clear fractions by multiplying by 5
$\dfrac{28(\mathbf{5})}{5} - \dfrac{12y(\mathbf{5})}{5} + 4y(\mathbf{5}) = 12(\mathbf{5})$	Reduce fractions and multiply
$28 - 12y + 20y = 60$	Combine like terms $-12y + 20y$
$28 + 8y = 60$	Subtract 28 from both sides
$\underline{\boldsymbol{-28}\qquad \boldsymbol{-28}}$	
$\dfrac{8y}{\mathbf{8}} = \dfrac{32}{\mathbf{8}}$	Divide both sides by 8
$y = 4$	We have our y
$x = \dfrac{-14}{5} + \dfrac{6(4)}{5}$	Plug into lone variable equation, multiply
$x = \dfrac{-14}{5} + \dfrac{24}{5}$	Add fractions
$x = \dfrac{10}{5}$	Reduce fraction
$x = 2$	Now we have our x
$(2, 4)$	Our Solution

Using the fractions does make the problem a bit more tricky. This is why we have another method for solving systems of equations that will be discussed in another lesson.

1.2 Practice - Substitution

Solve each system by substitution.

1) $y = -3x$
$y = 6x - 9$

2) $y = x + 5$
$y = -2x - 4$

3) $y = -2x - 9$
$y = 2x - 1$

4) $y = -6x + 3$
$y = 6x + 3$

5) $y = 6x + 4$
$y = -3x - 5$

6) $y = 3x + 13$
$y = -2x - 22$

7) $y = 3x + 2$
$y = -3x + 8$

8) $y = -2x - 9$
$y = -5x - 21$

9) $y = 2x - 3$
$y = -2x + 9$

10) $y = 7x - 24$
$y = -3x + 16$

11) $y = 6x - 6$
$-3x - 3y = -24$

12) $-x + 3y = 12$
$y = 6x + 21$

13) $y = -6$
$3x - 6y = 30$

14) $6x - 4y = -8$
$y = -6x + 2$

15) $y = -5$
$3x + 4y = -17$

16) $7x + 2y = -7$
$y = 5x + 5$

17) $-2x + 2y = 18$
$y = 7x + 15$

18) $y = x + 4$
$3x - 4y = -19$

19) $y = -8x + 19$
$-x + 6y = 16$

20) $y = -2x + 8$
$-7x - 6y = -8$

21) $7x - 2y = -7$
$y = 7$

22) $x - 2y = -13$
$4x + 2y = 18$

23) $x - 5y = 7$
$2x + 7y = -20$

24) $3x - 4y = 15$
$7x + y = 4$

25) $-2x - y = -5$
$x - 8y = -23$

26) $6x + 4y = 16$
$-2x + y = -3$

27) $-6x + y = 20$
$-3x - 3y = -18$

28) $7x + 5y = -13$
$x - 4y = -16$

29) $3x + y = 9$
$2x + 8y = -16$

30) $-5x - 5y = -20$
$-2x + y = 7$

31) $2x + y = 2$
$3x + 7y = 14$

32) $2x + y = -7$
$5x + 3y = -21$

33) $x + 5y = 15$
$-3x + 2y = 6$

34) $2x + 3y = -10$
$7x + y = 3$

35) $-2x + 4y = -16$
$y = -2$

36) $-2x + 2y = -22$
$-5x - 7y = -19$

37) $-6x + 6y = -12$
$8x - 3y = 16$

38) $-8x + 2y = -6$
$-2x + 3y = 11$

39) $2x + 3y = 16$
$-7x - y = 20$

40) $-x - 4y = -14$
$-6x + 8y = 12$

Answers - Substitution

1) $(1, -3)$
2) $(-3, 2)$
3) $(-2, -5)$
4) $(0, 3)$
5) $(-1, -2)$
6) $(-7, -8)$
7) $(1, 5)$
8) $(-4, -1)$
9) $(3, 3)$
10) $(4, 4)$
11) $(2, 6)$
12) $(-3, 3)$
13) $(-2, -6)$
14) $(0, 2)$
15) $(1, -5)$
16) $(-1, 0)$
17) $(-1, 8)$
18) $(3, 7)$
19) $(2, 3)$
20) $(8, -8)$
21) $(1, 7)$
22) $(1, 7)$
23) $(-3, -2)$
24) $(1, -3)$
25) $(1, 3)$
26) $(2, 1)$
27) $(-2, 8)$
28) $(-4, 3)$
29) $(4, -3)$
30) $(-1, 5)$
31) $(0, 2)$
32) $(0, -7)$
33) $(0, 3)$
34) $(1, -4)$
35) $(4, -2)$
36) $(8, -3)$
37) $(2, 0)$
38) $(2, 5)$
39) $(-4, 8)$
40) $(2, 3)$

1.3 Addition/Elimination

Objective: Solve systems of equations using the addition/elimination method.

When solving systems we have found that graphing is very limited when solving equations. We then considered a second method known as **substitution**. This is probably the most used idea in solving systems in various areas of algebra. How-ever, substitution can get ugly if we don't have a lone variable. This leads us to our second method for solving systems of equations. This method is known as either Elimination or Addition. We will set up the process in the following exam-ples, then define the five step process we can use to solve by elimination.

Example 1.

$3x - 4y = 8$ $\underline{5x + 4y = -24}$	Notice opposites in front of $y's$. Add columns.
$8x \quad = -16$	Solve for x, divide by 8
$\overline{8} \quad \overline{8}$	
$x = -2$	We have our x!
$5(-2) + 4y = -24$	Plug into either original equation, simplify
$-10 + 4y = -24$	Add 10 to both sides
$\underline{+10 \quad\quad +10}$	
$4y = -14$	Divide by 4
$\overline{4} \quad \overline{4}$	
$y = \frac{-7}{2}$	Now we have our y!
$\left(-2, \frac{-7}{2}\right)$	Our Solution

In the previous example one variable had opposites in front of it, $-4y$ and $4y$. Adding these together eliminated the y completely. This allowed us to solve for the x. This is the idea behind the addition method. However, generally we won't have opposites in front of one of the variables. In this case we will manipulate the equations to get the opposites we want by multiplying one or both equations (on both sides!). This is shown in the next example.

Example 2.

$-6x + 5y = 22$ $2x + 3y = 2$	We can get opposites in front of x, by multiplying the second equation by 3, to get $-6x$ and $+6x$

$3(2x + 3y) = (2)3$	Distribute to get new second equation.

$$
\begin{array}{rl}
6x+9y=6 & \text{New second equation}\\
\underline{-6x+5y=22} & \text{First equation still the same, add}\\
14y=28 & \text{Divide both sides by } 14\\
\overline{14}\quad\overline{14} & \\
y=2 & \text{We have our } y!\\
2x+3(2)=2 & \text{Plug into one of the original equations, simplify}\\
2x+6=2 & \text{Subtract 6 from both sides}\\
\underline{-6-6} & \\
2x=-4 & \text{Divide both sides by 2}\\
\overline{2}\quad\overline{2} & \\
x=-2 & \text{We also have our } x!\\
(-2,2) & \text{Our Solution}
\end{array}
$$

When we looked at the x terms, $-6x$ and $2x$ we decided to multiply the $2x$ by 3 to get the opposites we were looking for. What we are looking for with our opposites is the least common multiple (LCM) of the coefficients. We also could have solved the above problem by looking at the terms with y, $5y$ and $3y$. The LCM of 3 and 5 is 15. So we would want to multiply both equations, the $5y$ by 3, and the $3y$ by -5 to get opposites, $15y$ and $-15y$. This illustrates an important point, some problems we will have to multiply both equations by a constant (on both sides) to get the opposites we want.

Example 3.

$$
\begin{array}{rl}
3x+6y=-9 & \text{We can get opposites in front of } x\text{, find LCM of 6 and 9,}\\
2x+9y=-26 & \text{The LCM is 18. We will multiply to get } 18y \text{ and } -18y
\end{array}
$$

$$
\begin{array}{rl}
3(3x+6y)=(-9)3 & \text{Multiply the first equation by 3, both sides!}\\
9x+18y=-27 &
\end{array}
$$

$$
\begin{array}{rl}
-2(2x+9y)=(-26)(-2) & \text{Multiply the second equation by } -2\text{, both sides!}\\
-4x-18y=52 &
\end{array}
$$

$$
\begin{array}{rl}
9x+18y=-27 & \text{Add two new equations together}\\
\underline{-4x-18y=52} & \\
5x\qquad=25 & \text{Divide both sides by 5}\\
\overline{5}\qquad\quad\overline{5} & \\
x=5 & \text{We have our solution for } x\\
3(5)+6y=-9 & \text{Plug into either original equation, simplify}\\
15+6y=-9 & \text{Subtract 15 from both sides}\\
\underline{-15\qquad-15} &
\end{array}
$$

$$
\begin{aligned}
\frac{6y}{6} &= \frac{-24}{6} && \text{Divide both sides by 6}\\
y &= -4 && \text{Now we have our solution for } y\\
(5,-4) & && \text{Our Solution}
\end{aligned}
$$

It is important for each problem as we get started that all variables and constants are lined up before we start multiplying and adding equations. This is illustrated in the next example which includes the five steps we will go through to solve a problem using elimination.

Problem	$2x-5y=-13$ $-3y+4=-5x$
1. Line up the variables and constants	Second Equation: $-3y+4=-5x$ $+5x-4 \quad +5x-4$ $5x-3y=-4$
2. Multiply to get opposites (use LCD)	$2x-5y=-13$ $5x-3y=-4$ First Equation: multiply by -5 $-5(2x-5y)=(-13)(-5)$ $-10x+25y=65$ Second Equation: multiply by 2 $2(5x-3y)=(-4)2$ $10x-6y=-8$ $-10x+25y=65$ $10x-6y=-8$
3. Add	$19y=57$
4. Solve	$\frac{19y}{19}=\frac{57}{19}$ $y=3$
5. Plug into either original and solve	$2x-5(3)=-13$ $2x-15=-13$ $+15 \quad +15$ $\frac{2x}{2}=\frac{2}{2}$ $x=1$
Solution	$(1,3)$

World View Note: The famous mathematical text, *The Nine Chapters on the Mathematical Art*, which was printed around 179 AD in China describes a formula very similar to Gaussian elimination which is very similar to the addition method.

Just as with graphing and substution, it is possible to have no solution or infinite solutions with elimination. Just as with substitution, if the variables all disappear from our problem, a true statment will indicate infinite solutions and a false statment will indicate no solution.

Example 4.

$$\begin{array}{rl} 2x-5y=3 & \\ -6x+15y=-9 & \text{To get opposites in front of } x\text{, multiply first equation by } 3 \end{array}$$

$$\begin{array}{rl} 3(2x-5y)=(3)3 & \text{Distribute} \\ 6x-15y=9 & \end{array}$$

$$\begin{array}{rl} 6x-15y=9 & \text{Add equations together} \\ \underline{-6x+15y=-9} & \\ 0=0 & \text{True statement} \\ \text{Infinite solutions} & \text{Our Solution} \end{array}$$

Example 5.

$$\begin{array}{rl} 4x-6y=8 & \\ 6x-9y=15 & \text{LCM for } x's \text{ is } 12. \end{array}$$

$$\begin{array}{rl} 3(4x-6y)=(8)3 & \text{Multiply first equation by } 3 \\ 12x-18y=24 & \end{array}$$

$$\begin{array}{rl} -2(6x-9y)=(15)(-2) & \text{Multiply second equation by } -2 \\ -12x+18y=-30 & \end{array}$$

$$\begin{array}{rl} 12x-18y=24 & \text{Add both new equations together} \\ \underline{-12x+18y=-30} & \\ 0=-6 & \text{False statement} \\ \text{No Solution} & \text{Our Solution} \end{array}$$

We have covered three different methods that can be used to solve a system of two equations with two variables. While all three can be used to solve any system, graphing works great for small integer solutions. Substitution works great when we have a lone variable, and addition works great when the other two methods fail. As each method has its own strengths, it is important you are familiar with all three methods.

1.3 Practice - Addition/Elimination

Solve each system by elimination.

1) $4x+2y=0$
$-4x-9y=-28$

3) $-9x+5y=-22$
$9x-5y=13$

5) $-6x+9y=3$
$6x-9y=-9$

7) $4x-6y=-10$
$4x-6y=-14$

9) $-x-5y=28$
$-x+4y=-17$

11) $2x-y=5$
$5x+2y=-28$

13) $10x+6y=24$
$-6x+y=4$

15) $2x+4y=24$
$4x-12y=8$

17) $-7x+4y=-4$
$10x-8y=-8$

19) $5x+10y=20$
$-6x-5y=-3$

21) $-7x-3y=12$
$-6x-5y=20$

23) $9x-2y=-18$
$5x-7y=-10$

25) $9x+6y=-21$
$-10x-9y=28$

27) $-7x+5y=-8$
$-3x-3y=12$

29) $-8x-8y=-8$
$10x+9y=1$

31) $9y=7-x$
$-18y+4x=-26$

33) $0=9x+5y$
$y=\frac{2}{7}x$

2) $-7x+y=-10$
$-9x-y=-22$

4) $-x-2y=-7$
$x+2y=7$

6) $5x-5y=-15$
$5x-5y=-15$

8) $-3x+3y=-12$
$-3x+9y=-24$

10) $-10x-5y=0$
$-10x-10y=-30$

12) $-5x+6y=-17$
$x-2y=5$

14) $x+3y=-1$
$10x+6y=-10$

16) $-6x+4y=12$
$12x+6y=18$

18) $-6x+4y=4$
$-3x-y=26$

20) $-9x-5y=-19$
$3x-7y=-11$

22) $-5x+4y=4$
$-7x-10y=-10$

24) $3x+7y=-8$
$4x+6y=-4$

26) $-4x-5y=12$
$-10x+6y=30$

28) $8x+7y=-24$
$6x+3y=-18$

30) $-7x+10y=13$
$4x+9y=22$

32) $0=-9x-21+12y$
$1+\frac{4}{3}y+\frac{7}{3}x=0$

34) $-6-42y=-12x$
$x-\frac{1}{2}-\frac{7}{2}y=0$

Answers - Addition/Elimination

1) $(-2,4)$
2) $(2,4)$
3) No solution
4) Infinite number of solutions
5) No solution
6) Infinite number of solutions
7) No solution
8) $(2,-2)$
9) $(-3,-5)$
10) $(-3,6)$
11) $(-2,-9)$
12) $(1,-2)$
13) $(0,4)$
14) $(-1,0)$
15) $(8,2)$
16) $(0,3)$
17) $(4,6)$
18) $(-6,-8)$
19) $(-2,3)$
20) $(1,2)$
21) $(0,-4)$
22) $(0,1)$
23) $(-2,0)$
24) $(2,-2)$
25) $(-1,-2)$
26) $(-3,0)$
27) $(-1,-3)$
28) $(-3,0)$
29) $(-8,9)$
30) $(1,2)$
31) $(-2,1)$
32) $(-1,1)$
33) $(0,0)$
34) Infinite number of solutions

1.4 Value Problems

Objective: Solve value problems by setting up a system of equations.

One application of system of equations are known as value problems. Value problems are ones where each variable has a value attached to it. For example, if our variable is the number of nickles in a person's pocket, those nickles would have a value of five cents each. We will use a table to help us set up and solve value problems. The basic structure of the table is shown below.

	Number	Value	Total
Item 1			
Item 2			
Total			

The first column in the table is used for the number of things we have. Quite often, this will be our variables. The second column is used for the that value each item has. The third column is used for the total value which we calculate by multiplying the number by the value. For example, if we have 7 dimes, each with a value of 10 cents, the total value is $7 \cdot 10 = 70$ cents. The last row of the table is for totals. We only will use the third row (also marked total) for the totals that are given to use. This means sometimes this row may have some blanks in it. Once the table is filled in we can easily make equations by adding each column, setting it equal to the total at the bottom of the column. This is shown in the following example.

Example 1.

In a child's bank are 11 coins that have a value of \$1.85. The coins are either quarters or dimes. How many coins each does child have?

	Number	Value	Total
Quarter	q	25	
Dime	d	10	
Total			

Using value table, use q for quarters, d for dimes
Each quarter's value is 25 cents, dime's is 10 cents

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total			

Multiply number by value to get totals

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total	11		185

We have 11 coins total. This is the number total.
We have 1.85 for the final total,
Write final total in cents (185)
Because 25 and 10 are cents

$$q+d=11$$
$$25q+10d=185$$

First and last columns are our equations by adding
Solve by either addition or substitution.

$$-10(q+d)=(11)(-10)$$
$$-10q-10d=-110$$

Using addition, multiply first equation by -10

$$-10q-10d=-110$$
$$\underline{25q+10d=185}$$
$$15q\quad=75$$

Add together equations

Divide both sides by 15

$$\frac{15q}{15}=\frac{75}{15}$$

$$q=5$$

We have our q, number of quarters is 5

$$(5)+d=11$$
$$\underline{-5\qquad -5}$$
$$d=6$$

Plug into one of original equations
Subtract 5 from both sides
We have our d, number of dimes is 6

5 quarters and 6 dimes   Our Solution

World View Note: American coins are the only coins that do not state the value of the coin. On the back of the dime it says "one dime" (not 10 cents). On the back of the quarter it says "one quarter" (not 25 cents). On the penny it says "one cent" (not 1 cent). The rest of the world (Euros, Yen, Pesos, etc) all write the value as a number so people who don't speak the language can easily use the coins.

Ticket sales also have a value. Often different types of tickets sell for different prices (values). These problems can be solve in much the same way.

Example 2.

There were 41 tickets sold for an event. Tickets for children cost \$1.50 and tickets for adults cost \$2.00. Total receipts for the event were \$73.50. How many of each type of ticket were sold?

	Number	Value	Total
Child	c	1.5	
Adult	a	2	
Total			

Using our value table, c for child, a for adult
Child tickets have value 1.50, adult value is 2.00
(we can drop the zeros after the decimal point)

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total			

Multiply number by value to get totals

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total	41		73.5

We have 41 tickets sold. This is our number total
The final total was 73.50
Write in dollars as 1.5 and 2 are also dollars

$$c + a = 41$$ First and last columns are our equations by adding

$$1.5c + 2a = 73.5$$ We can solve by either addition or substitution

$$c + a = 41$$ We will solve by substitution.

$$\underline{-c \qquad -c}$$ Solve for a by subtracting c

$$a = 41 - c$$

$$1.5c + 2(41 - c) = 73.5$$ Substitute into untouched equation

$$1.5c + 82 - 2c = 73.5$$ Distribute

$$-0.5c + 82 = 73.5$$ Combine like terms

$$\underline{-82 \quad -82}$$ Subtract 82 from both sides

$$-0.5c = -8.5$$ Divide both sides by -0.5

$$\overline{-0.5} \quad \overline{-0.5}$$

$c = 17$	We have c, number of child tickets is 17
$a = 41 - (17)$	Plug into $a =$ equation to find a
$a = 24$	We have our a, number of adult tickets is 24
17 child tickets and 24 adult tickets	Our Solution

Some problems will not give us the total number of items we have. Instead they will give a relationship between the items. Here we will have statements such as "There are twice as many dimes as nickles". While it is clear that we need to multiply one variable by 2, it may not be clear which variable gets multiplied by 2. Generally the equations are backwards from the English sentence. If there are twice as many dimes, than we multiply the other variable (nickels) by two. So the equation would be $d = 2n$. This type of problem is in the next example.

Example 3.

A man has a collection of stamps made up of 5 cent stamps and 8 cent stamps. There are three times as many 8 cent stamps as 5 cent stamps. The total value of all the stamps is $3.48. How many of each stamp does he have?

	Number	Value	Total
Five	f	5	$5f$
Eight	$3f$	8	$24f$
Total			348

Use value table, f for five cent stamp, and e for eight
Also list value of each stamp under value column

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			

Multiply number by value to get total

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			348

The final total was 338 (written in cents)
We do not know the total number, this is left blank.

$e = 3f$	**3 times as many 8 cent stamples as 5 cent stamps**
$5f + 8e = 348$	Total column gives second equation
$5f + 8(3f) = 348$	Substitution, substitute first equation in second
$5f + 24f = 348$	Multiply first
$29f = 348$	Combine like terms
$\overline{29} \quad \overline{29}$	Divide both sides by 39
$f = 12$	We have f. There are 12 five cent stamps
$e = 3(12)$	Plug into first equation

$e = 36$	We have e, There are 36 eight cent stamps
12 five cent, 36 eight cent stamps	Our Solution

The same process for solving value problems can be applied to solving interest problems. Our table titles will be adjusted slightly as we do so.

	Invest	Rate	Interest
Account 1			
Account 2			
Total			

Our first column is for the amount invested in each account. The second column is the interest rate earned (written as a decimal - move decimal point twice left), and the last column is for the amount of interset earned. Just as before, we multiply the investment amount by the rate to find the final column, the interest earned. This is shown in the following example.

Example 4.

A woman invests \$4000 in two accounts, one at 6% interest, the other at 9% interest for one year. At the end of the year she had earned \$270 in interest. How much did she have invested in each account?

	Invest	Rate	Interest
Account 1	x	0.06	
Account 2	y	0.09	
Total			

Use our investment table, x and y for accounts
Fill in interest rates as decimals

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total			

Multiply across to find interest earned.

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total	4000		270

Total investment is 4000,
Total interest was 276

$x + y = 4000$	First and last column give our two equations
$0.06x + 0.09y = 270$	Solve by either substitution or addition
$-0.06(x + y) = (4000)(-0.06)$	Use Addition, multiply first equation by -0.06
$-0.06x - 0.06y = -240$	
$-0.06x - 0.06y = -240$	Add equations together
$\underline{0.06x + 0.09y = 270}$	

$$\frac{0.03y = 30}{0.03 \quad 0.03}$$ Divide both sides by 0.03

$y = 1000$ We have y, \$1000 invested at 9%

$x + 1000 = 4000$ Plug into original equation

$-1000 - 1000$ Subtract 1000 from both sides

$x = 3000$ We have x, \$3000 invested at 6%

\$1000 at 9% and \$3000 at 6% Our Solution

The same process can be used to find an unknown interest rate.

Example 5.

John invests \$5000 in one account and \$8000 in an account paying 4% more in interest. He earned \$1230 in interest after one year. At what rates did he invest?

	Invest	Rate	Interest
Account 1	5000	x	
Account 2	8000	$x + 0.04$	
Total			

Our investment table. Use x for first rate
The second rate is 4% higher, or $x + 0.04$
Be sure to write this rate as *a* decimal!

	Invest	Rate	Interest
Account 1	5000	x	$5000x$
Account 2	8000	$x + 0.04$	$8000x + 320$
Total			

Multiply to fill in interest column.
Be sure to distribute $8000(x + 0.04)$

	Invest	Rate	Interest
Account 2	5000	x	$5000x$
Account 2	8000	$x + 0.04$	$8000x + 320$
Total			1230

Total interest was 1230.

$5000x + 8000x + 320 = 1230$ Last column gives our equation

$13000x + 320 = 1230$ Combine like terms

$\underline{-320 \quad -320}$ Subtract 320 from both sides

$$\frac{13000x = 910}{13000 \quad 13000}$$ Divide both sides by 13000

$x = 0.07$ We have our x, 7% interest

$(0.07) + 0.04$ Second account is 4% higher

0.11 The account with \$8000 is at 11%

\$5000 at 7% and \$8000 at 11% Our Solution

1.4 Practice - Value Problems

Solve.

1) A collection of dimes and quaters is worth \$15.25. There are 103 coins in all. How many of each is there?

2) A collection of half dollars and nickels is worth \$13.40. There are 34 coins in all. How many are there?

3) The attendance at a school concert was 578. Admission was \$2.00 for adults and \$1.50 for children. The total receipts were \$985.00. How many adults and how many children attended?

4) A purse contains \$3.90 made up of dimes and quarters. If there are 21 coins in all, how many dimes and how many quarters were there?

5) A boy has \$2.25 in nickels and dimes. If there are twice as many dimes as nickels, how many of each kind has he?

6) \$3.75 is made up of quarters and half dollars. If the number of quarters exceeds the number of half dollars by 3, how many coins of each denomination are there?

7) A collection of 27 coins consisting of nickels and dimes amounts to \$2.25. How many coins of each kind are there?

8) \$3.25 in dimes and nickels, were distributed among 45 boys. If each received one coin, how many received dimes and how many received nickels?

9) There were 429 people at a play. Admission was \$1 each for adults and 75 cents each for children. The receipts were \$372.50. How many children and how many adults attended?

10) There were 200 tickets sold for a women's basketball game. Tickets for students were 50 cents each and for adults 75 cents each. The total amount of money collected was \$132.50. How many of each type of ticket was sold?

11) There were 203 tickets sold for a volleyball game. For activity-card holders, the price was \$1.25 each and for noncard holders the price was \$2 each. The total amount of money collected was \$310. How many of each type of ticket was sold?

12) At a local ball game the hotdogs sold for \$2.50 each and the hamburgers sold for \$2.75 each. There were 131 total sandwiches sold for a total value of \$342. How many of each sandwich was sold?

13) At a recent Vikings game \$445 in admission tickets was taken in. The cost of a student ticket was \$1.50 and the cost of a non-student ticket was \$2.50. A total of 232 tickets were sold. How many students and how many non-students attented the game?

14) A bank contains 27 coins in dimes and quarters. The coins have a total value of \$4.95. Find the number of dimes and quarters in the bank.

15) A coin purse contains 18 coins in nickels and dimes. The coins have a total value of \$1.15. Find the number of nickels and dimes in the coin purse.

16) A business executive bought 40 stamps for \$9.60. The purchase included 25¢ stamps and 20¢ stamps. How many of each type of stamp were bought?

17) A postal clerk sold some 15¢ stamps and some 25¢ stamps. Altogether, 15 stamps were sold for a total cost of \$3.15. How many of each type of stamps were sold?

18) A drawer contains 15¢ stamps and 18¢ stamps. The number of 15¢ stamps is four less than three times the number of 18¢ stamps. The total value of all the stamps is \$1.29. How many 15¢ stamps are in the drawer?

19) The total value of dimes and quarters in a bank is \$6.05. There are six more quarters than dimes. Find the number of each type of coin in the bank.

20) A child's piggy bank contains 44 coins in quarters and dimes. The coins have a total value of \$8.60. Find the number of quaters in the bank.

21) A coin bank contains nickels and dimes. The number of dimes is 10 less than twice the number of nickels. The total value of all the coins is \$2.75. Find the number of each type of coin in the bank.

22) A total of 26 bills are in a cash box. Some of the bills are one dollar bills, and the rest are five dollar bills. The total amount of cash in the box is \$50. Find the number of each type of bill in the cash box.

23) A bank teller cashed a check for \$200 using twenty dollar bills and ten dollar bills. In all, twelve bills were handed to the customer. Find the number of twenty dollar bills and the number of ten dollar bills.

24) A collection of stamps consists of 22¢ stamps and 40¢ stamps. The number of 22¢ stamps is three more than four times the number of 40¢ stamps. The total value of the stamps is \$8.34. Find the number of 22¢ stamps in the collection.

25) A total of \$27000 is invested, part of it at 12% and the rest at 13%. The total interest after one year is \$3385. How much was invested at each rate?

26) A total of \$50000 is invested, part of it at 5% and the rest at 7.5%. The total interest after one year is \$3250. How much was invested at each rate?

27) A total of \$9000 is invested, part of it at 10% and the rest at 12%. The total interest after one year is \$1030. How much was invested at each rate?

28) A total of \$18000 is invested, part of it at 6% and the rest at 9%. The total interest after one year is \$1248. How much was invested at each rate?

29) An inheritance of \$10000 is invested in 2 ways, part at 9.5% and the remainder at 11%. The combined annual interest was \$1038.50. How much was invested at each rate?

30) Kerry earned a total of \$900 last year on his investments. If \$7000 was invested at a certain rate of return and \$9000 was invested in a fund with a rate that was 2% higher, find the two rates of interest.

31) Jason earned \$256 interest last year on his investments. If \$1600 was invested at a certain rate of return and \$2400 was invested in a fund with a rate that was double the rate of the first fund, find the two rates of interest.

32) Millicent earned \$435 last year in interest. If \$3000 was invested at a certain rate of return and \$4500 was invested in a fund with a rate that was 2% lower, find the two rates of interest.

33) A total of \$8500 is invested, part of it at 6% and the rest at 3.5%. The total interest after one year is \$385. How much was invested at each rate?

34) A total of \$12000 was invested, part of it at 9% and the rest at 7.5%. The total interest after one year is \$1005. How much was invested at each rate?

35) A total of \$15000 is invested, part of it at 8% and the rest at 11%. The total interest after one year is \$1455. How much was invested at each rate?

36) A total of \$17500 is invested, part of it at 7.25% and the rest at 6.5%. The total interest after one year is \$1227.50. How much was invested at each rate?

37) A total of \$6000 is invested, part of it at 4.25% and the rest at 5.75%. The total interest after one year is \$300. How much was invested at each rate?

38) A total of \$14000 is invested, part of it at 5.5% and the rest at 9%. The total interest after one year is \$910. How much was invested at each rate?

39) A total of \$11000 is invested, part of it at 6.8% and the rest at 8.2%. The total interest after one year is \$797. How much was invested at each rate?

40) An investment portfolio earned \$2010 in interest last year. If \$3000 was invested at a certain rate of return and \$24000 was invested in a fund with a rate that was 4% lower, find the two rates of interest.

41) Samantha earned \$1480 in interest last year on her investments. If \$5000 was invested at a certain rate of return and \$11000 was invested in a fund with a rate that was two-thirds the rate of the first fund, find the two rates of interest.

42) A man has \$5.10 in nickels, dimes, and quarters. There are twice as many nickels as dimes and 3 more dimes than quarters. How many coins of each kind were there?

43) 30 coins having a value of \$3.30 consists of nickels, dimes and quarters. If there are twice as many quarters as dimes, how many coins of each kind were there?

44) A bag contains nickels, dimes and quarters having a value of \$3.75. If there are 40 coins in all and 3 times as many dimes as quarters, how many coins of each kind were there?

Answers - Value Problems

1) 33Q, 70D

2) 26 h, 8 n

3) 236 adult, 342 child

4) 9d, 12q

5) 9, 18

6) 7q, 4h

7) 9, 18

8) 25, 20

9) 203 adults, 226 child

10) 130 adults, 70 students

11) 128 card, 75 no card

12) 73 hotdogs, 58 hamburgers

13) 135 students, 97 non-students

14) 12d, 15q

15) 13n, 5d

16) 8 20¢, 32 25¢

17) 6 15¢, 9 25¢

18) 5

19) 13 d, 19 q

20) 28 q

21) 15 n, 20 d

22) 20 \$1, 6 \$5

23) 8 \$20, 4 \$10

24) 27

25) \$12500 @ 12%
\$14500 @ 13%

26) \$20000 @ 5%
\$30000 @ 7.5%

27) \$2500 @ 10%
\$6500 @ 12%

28) \$12400 @ 6%
\$5600 @ 9%

29) \$4100 @ 9.5%
\$5900 @ 11%

30) \$7000 @ 4.5%
\$9000 @ 6.5%

31) \$1600 @ 4%;
\$2400 @ 8%

32) \$3000 @ 4.6%
\$4500 @ 6.6%

33) \$3500 @ 6%;
\$5000 @ 3.5%

34) \$7000 @ 9%
\$5000 @ 7.5%

35) \$6500 @ 8%;
\$8500 @ 11%

36) \$12000 @ 7.25%
\$5500 @ 6.5%

37) \$3000 @ 4.25%;
\$3000 @ 5.75%

38) \$10000 @ 5.5%
\$4000 @ 9%

39) \$7500 @ 6.8%;
\$3500 @ 8.2%

40) \$3000 @ 11%;
\$24000 @ 7%

41) \$5000 @ 12%
\$11000 @ 8%

42) 26n, 13d, 10q

43) 18, 4, 8

44) 20n, 15d, 10q

1.5 Mixture Problems

Objective: Solve mixture problems by setting up a system of equations.

One application of systems of equations are mixture problems. Mixture problems are ones where two different solutions are mixed together resulting in a new final solution. We will use the following table to help us solve mixture problems:

	Amount	Part	Total
Item 1			
Item 2			
Final			

The first column is for the amount of each item we have. The second column is labeled "part". If we mix percentages we will put the rate (written as a decimal) in this column. If we mix prices we will put prices in this column. Then we can multiply the amount by the part to find the total. Then we can get an equation by adding the amount and/or total columns that will help us solve the problem and answer the questions.

These problems can have either one or two variables. We will start with one variable problems.

Example 1.

A chemist has 70 mL of a 50% methane solution. How much of a 80% solution must she add so the final solution is 60% methane?

	Amount	Part	Total
Start	70	0.5	
Add	x	0.8	
Final			

Set up the mixture table. We start with 70, but don't know how much we add, that is x. The part is the percentages, 0.5 for start, 0.8 for add.

	Amount	Part	Total
Start	70	0.5	
Add	x	0.8	
Final	$70+x$	0.6	

Add amount column to get final amount. The part for this amount is 0.6 because we want the final solution to be 60% methane.

	Amount	Part	Total
Start	70	0.5	35
Add	x	0.8	$0.8x$
Final	$70+x$	0.6	$42+0.6x$

Multiply amount by part to get total.
be sure to distribute on the last row: $(70+x)0.6$

$35+0.8x=42+0.6x$ The last column is our equation by adding

$-0.6x \quad -0.6x$ Move variables to one side, subtract $0.6x$

$35+0.2x=42$ Subtract 35 from both sides

$-35 \quad -35$

$0.2x=7$ Divide both sides by 0.2

$\overline{0.2} \quad \overline{0.2}$

$x=35$ We have our x!

35 mL must be added Our Solution

The same process can be used if the starting and final amount have a price attached to them, rather than a percentage.

Example 2.

A coffee mix is to be made that sells for \$2.50 by mixing two types of coffee. The cafe has 40 mL of coffee that costs \$3.00. How much of another coffee that costs \$1.50 should the cafe mix with the first?

	Amount	Part	Total
Start	40	3	
Add	x	1.5	
Final			

Set up mixture table. We know the starting amount and its cost, \$3. The added amount we do not know but we do know its cost is \$1.50.

	Amount	Part	Total
Start	40	3	
Add	x	1.5	
Final	$40+x$	2.5	

Add the amounts to get the final amount.
We want this final amount to sell for \$2.50.

	Amount	Part	Total
Start	40	3	120
Add	x	1.5	$1.5x$
Final	$40+x$	2.5	$100+2.5x$

Multiply amount by part to get the total.
Be sure to distribute on the last row $(40+x)2.5$

$$120+1.5x=100+2.5x$$ Adding down the total column gives our equation

$$-1.5x \qquad -1.5x$$ Move variables to one side by subtracting $1.5x$

$$120=100+x$$ Subtract 100 from both sides

$$-100 \quad -100$$

$$20=x$$ We have our x.

20mL must be added. Our Solution

World View Note: Brazil is the world's largest coffee producer, producing 2.59 million metric tons of coffee a year! That is over three times as much coffee as second place Vietnam!

The above problems illustrate how we can put the mixture table together and get an equation to solve. However, here we are interested in systems of equations, with two unknown values. The following example is one such problem.

Example 3.

A farmer has two types of milk, one that is 24% butterfat and another which is 18% butterfat. How much of each should he use to end up with 42 gallons of 20% butterfat?

	Amount	Part	Total
Milk 1	x	0.24	
Milk 2	y	0.18	
Final	42	0.2	

We don't know either start value, but we do know final is 42. Also fill in part column with percentage of each type of milk including the final solution

	Amount	Part	Total
Milk 1	x	0.24	$0.24x$
Milk 2	y	0.18	$0.18y$
Final	42	0.2	8.4

Multiply amount by part to get totals.

$$x+y=42$$ The amount column gives one equation

$$0.24x+0.18y=8.4$$ The total column gives a second equation.

$$-0.18(x+y)=(42)(-0.18)$$ Use addition. Multiply first equation by -0.18

$$-0.18x-0.18y=-7.56$$

$-0.18x-0.18y=-7.56$	Add the equations together
$\underline{0.24x+0.18y=8.4}$	
$\frac{0.06x}{0.06} = \frac{0.84}{0.06}$	Divide both sides by 0.06
$x=14$	We have our x, 14 gal of 24% butterfat
$(14)+y=42$	Plug into original equation to find y
$\underline{-14 \quad -14}$	Subtract 14 from both sides
$y=28$	We have our y, 28 gal of 18% butterfat
14 gal of 24% and 28 gal of 18%	Our Solution

The same process can be used to solve mixtures of prices with two unknowns.

Example 4.

In a candy shop, chocolate which sells for \$4 a pound is mixed with nuts which are sold for \$2.50 a pound are mixed to form a chocolate-nut candy which sells for \$3.50 a pound. How much of each are used to make 30 pounds of the mixture?

	Amount	Part	Total
Chocolate	c	4	
Nut	n	2.5	
Final	30	3.5	

Using our mixture table, use c and n for variables
We do know the final amount (30) and price, include this in the table

	Amount	Part	Total
Chocolate	c	4	$4c$
Nut	n	2.5	$2.5n$
Final	30	3.5	105

Multiply amount by part to get totals

$c+n=30$	First equation comes from the first column
$4c+2.5n=105$	Second equation comes from the total column

$c+n=30$	We will solve this problem with substitution
$-n \quad -n$	Solve for c by subtracting n from the first equation
$c=30-n$	

$4(30-n)+2.5n=105$	Substitute into untouched equation
$120-4n+2.5n=105$	Distribute

$120 - 1.5n = 105$	Combine like terms
$\underline{-120 \qquad -120}$	Subtract 120 from both sides
$\frac{-1.5n}{-1.5} = \frac{-15}{-1.5}$	Divide both sides by -1.5
$n = 10$	We have our n, 10 lbs of nuts
$c = 30 - (10)$	Plug into $c =$ equation to find c
$c = 20$	We have our c, 20 lbs of chocolate
10 lbs of nuts and 20 lbs of chocolate	Our Solution

With mixture problems we often are mixing with a pure solution or using water which contains none of our chemical we are interested in. For pure solutions, the percentage is 100% (or 1 in the table). For water, the percentage is 0%. This is shown in the following example.

Example 5.

A solution of pure antifreeze is mixed with water to make a 65% antifreeze solution. How much of each should be used to make 70 L?

	Amount	Part	Final
Antifreeze	a	1	
Water	w	0	
Final	70	0.65	

We use a and w for our variables. Antifreeze is pure, 100% or 1 in our table, written as a decimal. Water has no antifreeze, its percentage is 0. We also fill in the final percent

	Amount	Part	Final
Antifreeze	a	1	a
Water	w	0	0
Final	70	0.65	45.5

Multiply to find final amounts

$a + w = 70$	First equation comes from first column
$a = 45.5$	Second equation comes from second column
$(45.5) + w = 70$	We have a, plug into to other equation
$-45.5 \qquad -45.5$	Subtract 45.5 from both sides
$w = 24.5$	We have our w
$45.5\,L$ of antifreeze and $24.5\,L$ of water	Our Solution

1.5 Practice - Mixture Problems

Solve.

1) A tank contains 8000 liters of a solution that is 40% acid. How much water should be added to make a solution that is 30% acid?

2) How much antifreeze should be added to 5 quarts of a 30% mixture of antifreeze to make a solution that is 50% antifreeze?

3) Of 12 pounds of salt water 10% is salt; of another mixture 3% is salt. How many pounds of the second should be added to the first in order to get a mixture of 5% salt?

4) How much alcohol must be added to 24 gallons of a 14% solution of alcohol in order to produce a 20% solution?

5) How many pounds of a 4% solution of borax must be added to 24 pounds of a 12% solution of borax to obtain a 10% solution of borax?

6) How many grams of pure acid must be added to 40 grams of a 20% acid solution to make a solution which is 36% acid?

7) A 100 LB bag of animal feed is 40% oats. How many pounds of oats must be added to this feed to produce a mixture which is 50% oats?

8) A 20 oz alloy of platinum that costs $220 per ounce is mixed with an alloy that costs $400 per ounce. How many ounces of the $400 alloy should be used to make an alloy that costs $300 per ounce?

9) How many pounds of tea that cost $4.20 per pound must be mixed with 12 lb of tea that cost $2.25 per pound to make a mixture that costs $3.40 per pound?

10) How many liters of a solvent that costs $80 per liter must be mixed with 6 L of a solvent that costs $25 per liter to make a solvent that costs $36 per liter?

11) How many kilograms of hard candy that cost $7.50 per kilogram must be mixed with 24 kg of jelly beans that cost $3.25 per kilogram to make a mixture that sells for $4.50 per kilogram?

12) How many kilograms of soil supplement that costs $7.00 per kilogram must be mixed with 20 kg of aluminum nitrate that costs $3.50 per kilogram to make a fertilizer that costs $4.50 per kilogram?

13) How many pounds of lima beans that cost 90¢ per pound must be mixed with 16 lb of corn that cost 50¢ per pound to make a mixture of vegetables that costs 65¢ per pound?

14) How many liters of a blue dye that costs $1.60 per liter must be mixed with 18 L of anil that costs $2.50 per liter to make a mixture that costs $1.90 per liter?

15) Solution A is 50% acid and solution B is 80% acid. How much of each should be used to make 100cc. of a solution that is 68% acid?

16) A certain grade of milk contains 10% butter fat and a certain grade of cream

60% butter fat. How many quarts of each must be taken so as to obtain a mixture of 100 quarts that will be 45% butter fat?

17) A farmer has some cream which is 21% butterfat and some which is 15% butter fat. How many gallons of each must be mixed to produce 60 gallons of cream which is 19% butterfat?

18) A syrup manufacturer has some pure maple syrup and some which is 85% maple syrup. How many liters of each should be mixed to make 150L which is 96% maple syrup?

19) A chemist wants to make 50ml of a 16% acid solution by mixing a 13% acid solution and an 18% acid solution. How many milliliters of each solution should the chemist use?

20) A hair dye is made by blending 7% hydrogen peroxide solution and a 4% hydrogen peroxide solution. How many mililiters of each are used to make a 300 ml solution that is 5% hydrogen peroxide?

21) A paint that contains 21% green dye is mixed with a paint that contains 15% green dye. How many gallons of each must be used to make 60 gal of paint that is 19% green dye?

22) A candy mix sells for $2.20 per kilogram. It contains chocolates worth $1.80 per kilogram and other candy worth $3.00 per kilogram. How much of each are in 15 kilograms of the mixture?

23) To make a weed and feed mixture, the Green Thumb Garden Shop mixes fertilizer worth $4.00/lb. with a weed killer worth $8.00/lb. The mixture will cost $6.00/lb. How much of each should be used to prepare 500 lb. of the mixture?

24) A grocer is mixing 40 cent per lb. coffee with 60 cent per lb. coffee to make a mixture worth 54¢ per lb. How much of each kind of coffee should be used to make 70 lb. of the mixture?

25) A grocer wishes to mix sugar at 9 cents per pound with sugar at 6 cents per pound to make 60 pounds at 7 cents per pound. What quantity of each must he take?

26) A high-protein diet supplement that costs $6.75 per pound is mixed with a vitamin supplement that costs $3.25 per pound. How many pounds of each should be used to make 5 lb of a mixture that costs $4.65 per pound?

27) A goldsmith combined an alloy that costs $4.30 per ounce with an alloy that costs $1.80 per ounce. How many ounces of each were used to make a mixture of 200 oz costing $2.50 per ounce?

28) A grocery store offers a cheese and fruit sampler that combines cheddar cheese that costs $8 per kilogram with kiwis that cost $3 per kilogram. How many kilograms of each were used to make a 5 kg mixture that costs $4.50 per kilogram?

29) The manager of a garden shop mixes grass seed that is 60% rye grass with 70 lb of grass seed that is 80% rye grass to make a mixture that is 74% rye grass. How much of the 60% mixture is used?

30) How many ounces of water evaporated from 50 oz of a 12% salt solution to produce a 15% salt solution?

31) A caterer made an ice cream punch by combining fruit juice that cost $2.25 per gallon with ice cream that costs $3.25 per gallon. How many gallons of each were used to make 100 gal of punch costing $2.50 per pound?

32) A clothing manufacturer has some pure silk thread and some thread that is 85% silk. How many kilograms of each must be woven together to make 75 kg of cloth that is 96% silk?

33) A carpet manufacturer blends two fibers, one 20% wool and the second 50% wool. How many pounds of each fiber should be woven together to produce 600 lb of a fabric that is 28% wool?

34) How many pounds of coffee that is 40% java beans must be mixed with 80 lb of coffee that is 30% java beans to make a coffee blend that is 32% java beans?

35) The manager of a specialty food store combined almonds that cost $4.50 per pound with walnuts that cost $2.50 per pound. How many pounds of each were used to make a 100 lb mixture that cost $3.24 per pound?

36) A tea that is 20% jasmine is blended with a tea that is 15% jasmine. How many pounds of each tea are used to make 5 lb of tea that is 18% jasmine?

37) How many ounces of dried apricots must be added to 18 oz of a snack mix that contains 20% dried apricots to make a mixture that is 25% dried apricots?

38) How many mililiters of pure chocolate must be added to 150 ml of chocolate topping that is 50% chocolate to make a topping that is 75% chocolate?

39) How many ounces of pure bran flakes must be added to 50 oz of cereal that is 40% bran flakes to produce a mixture that is 50% bran flakes?

40) A ground meat mixture is formed by combining meat that costs $2.20 per pound with meat that costs $4.20 per pound. How many pounds of each were used to make a 50 lb mixture tha costs $3.00 per pound?

41) How many grams of pure water must be added to 50 g of pure acid to make a solution that is 40% acid?

42) A lumber company combined oak wood chips that cost $3.10 per pound with pine wood chips that cost $2.50 per pound. How many pounds of each were used to make an 80 lb mixture costing $2.65 per pound?

43) How many ounces of pure water must be added to 50 oz of a 15% saline solution to make a saline solution that is 10% salt?

1.5

Answers - Mixture Problems

1) 2666.7

2) 2

3) 30

4) 1, 8

5) 8

6) 10

7) 20

8) 16

9) 17.25

10) 1.5

11) 10

12) 8

13) 9.6

14) 36

15) 40, 60

16) 30, 70

17) 40, 20

18) 40, 110

19) 20, 30

20) 100, 200

21) 40, 20

22) 10, 5

23) 250, 250

24) 21, 49

25) 20, 40

26) 2, 3

27) 56, 144

28) 1.5, 3.5

29) 30

30) 10

31) 75, 25

32) 55, 20

33) 440, 160

34) 20

35) 35, 63

36) 3, 2

37) 1.2

38) 150

39) 10

40) 30, 20

41) 75

42) 20, 60

43) 25

Chapter 2 - Polynomials
2.1 Exponent Properties

Objective: Simplify expressions using the properties of exponents.

Problems with expoenents can often be simplified using a few basic exponent properties. Exponents represent repeated multiplication. We will use this fact to discover the important properties.

World View Note: The word exponent comes from the Latin "expo" meaning out of and "ponere" meaning place. While there is some debate, it seems that the Babylonians living in Iraq were the first to do work with exponents (dating back to the 23rd century BC or earlier!)

Example 1.

$$\begin{array}{rl} a^3a^2 & \text{Expand exponents to multiplication problem} \\ (aaa)(aa) & \text{Now we have } 5\, a's \text{ being multiplied together} \\ a^5 & \text{Our Solution} \end{array}$$

A quicker method to arrive at our answer would have been to just add the exponents: $a^3a^2 = a^{3+2} = a^5$ This is known as the **product rule of exponents**

Product Rule of Exponents: $\boldsymbol{a^m a^n = a^{m+n}}$

The product rule of exponents can be used to simplify many problems. We will add the exponent on like variables. This is shown in the following examples

Example 2.

$$\begin{array}{rl} 3^2 \cdot 3^6 \cdot 3 & \text{Same base, add the exponents } 2+6+1 \\ 3^9 & \text{Our Solution} \end{array}$$

Example 3.

$$\begin{array}{rl} 2x^3y^5z \cdot 5xy^2z^3 & \text{Multiply } 2\cdot 5\text{, add exponents on } x, y \text{ and } z \\ 10x^4y^7z^4 & \text{Our Solution} \end{array}$$

Rather than multiplying, we will now try to divide with exponents

Example 4.

$$\begin{array}{rl} \dfrac{a^5}{a^2} & \text{Expand exponents} \\ \dfrac{aaaaa}{aa} & \text{Divide out two of the } a's \\ aaa & \text{Convert to exponents} \\ a^3 & \text{Our Solution} \end{array}$$

A quicker method to arrive at the solution would have been to just subtract the exponents, $\frac{a^5}{a^2} = a^{5-2} = a^3$. This is known as the quotient rule of exponents.

$$\textbf{Quotient Rule of Exponents: } \frac{a^m}{a^n} = a^{m-n}$$

The quotient rule of exponents can similarly be used to simplify exponent problems by subtracting exponents on like variables. This is shown in the following examples.

Example 5.

$\dfrac{7^{13}}{7^5}$	Same base, subtract the exponents
7^8	Our Solution

Example 6.

$\dfrac{5a^3b^5c^2}{2\mathrm{ab}^3c}$	Subtract exponents on a, b and c
$\dfrac{5}{2}a^2b^2c$	Our Solution

A third property we will look at will have an exponent problem raised to a second exponent. This is investigated in the following example.

Example 7.

$\left(a^2\right)^3$	This means we have a^2 three times
$a^2 \cdot a^2 \cdot a^2$	Add exponents
a^6	Our solution

A quicker method to arrive at the solution would have been to just multiply the exponents, $(a^2)^3 = a^{2\cdot 3} = a^6$. This is known as the power of a power rule of exponents.

$$\textbf{Power of } a \textbf{ Power Rule of Exponents: } (a^m)^n = a^{mn}$$

This property is often combined with two other properties which we will investigate now.

Example 8.

$(ab)^3$	This means we have (ab) three times
$(ab)(ab)(ab)$	Three $a's$ and three $b's$ can be written with exponents
a^3b^3	Our Solution

A quicker method to arrive at the solution would have been to take the exponent of three and put it on each factor in parenthesis, $(ab)^3 = a^3b^3$. This is known as the power of a product rule or exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Product Rule of Exponents: } \mathbf{(ab)}^{\boldsymbol{m}} = \boldsymbol{a^m b^m}$$

It is important to be careful to only use the power of a product rule with multiplication inside parenthesis. This property does NOT work if there is addition or subtraction.

Warning 9.

$(a+b)^m \neq a^m + b^m$ These are **NOT** equal, beware of this error!

Another property that is very similar to the power of a product rule is considered next.

Example 10.

$$\left(\frac{a}{b}\right)^3 \quad \text{This means we have the fraction three timse}$$

$$\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) \quad \text{Multiply fractions across the top and bottom, using exponents}$$

$$\frac{a^3}{b^3} \quad \text{Our Solution}$$

A quicker method to arrive at the solution would have been to put the exponent on every factor in both the numerator and denominator, $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$. This is known as the power of a quotient rule of exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Quotient Rule of Exponents: } \left(\boldsymbol{\frac{a}{b}}\right)^{\boldsymbol{m}} = \boldsymbol{\frac{a^m}{b^m}}$$

The power of a power, product and quotient rules are often used together to simplify expressions. This is shown in the following examples.

Example 11.

$$(x^3yz^2)^4 \quad \text{Put the exponent of 4 on each factor, multiplying powers}$$

$$x^{12}y^4z^8 \quad \text{Our solution}$$

Example 12.

$$\left(\frac{a^3b}{c^8d^5}\right)^2 \quad \text{Put the exponent of 2 on each factor, multiplying powers}$$

$$\frac{a^6b^2}{c^8d^{10}} \quad \text{Our Solution}$$

As we multiply exponents its important to remember these properties apply to exponents, not bases. An expressions such as 5^3 does not mean we multipy 5 by 3, rather we multiply 5 three times, $5 \times 5 \times 5 = 125$. This is shown in the next example.

Example 13.

$(4x^2y^5)^3$	Put the exponent of 3 on each factor, multiplying powers
$4^3x^6y^{15}$	Evaluate 4^3
$64x^6y^{15}$	Our Solution

In the previous example we did not put the 3 on the 4 and multipy to get 12, this would have been incorrect. Never multipy a base by the exponent. These properties pertain to exponents only, not bases.

In this lesson we have discussed 5 different exponent properties. These rules are summarized in the following table.

Rules of Exponents

Product Rule of Exponents	$a^ma^n = a^{m+n}$
Quotient Rule of Exponents	$\frac{a^m}{a^n} = a^{m-n}$
Power of a Power Rule of Exponents	$(a^m)^n = a^{mn}$
Power of a Product Rule of Exponents	$(ab)^m = a^mb^m$
Power of a Quotient Rule of Exponents	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

These five properties are often mixed up in the same problem. Often there is a bit of flexibility as to which property is used first. However, order of operations still applies to a problem. For this reason it is the suggestion of the auther to simplify inside any parenthesis first, then simplify any exponents (using power rules), and finally simplify any multiplication or division (using product and quotient rules). This is illustrated in the next few examples.

Example 14.

$(4x^3y \cdot 5x^4y^2)^3$	In parenthesis simplify using product rule, adding exponents
$(20x^7y^3)^3$	With power rules, put three on each factor, multiplying exponents
$20^3x^{21}y^9$	Evaluate 20^3
$8000x^{21}y^9$	Our Solution

Example 15.

$$\begin{array}{rl} 7a^3(2a^4)^3 & \text{Parenthesis are already simplified, next use power rules} \\ 7a^3(8a^{12}) & \text{Using product rule, add exponents and multiply numbers} \\ 56a^{15} & \text{Our Solution} \end{array}$$

Example 16.

$$\begin{array}{rl} \dfrac{3a^3b\cdot 10a^4b^3}{2a^4b^2} & \text{Simplify numerator with product rule, adding exponents} \\[2ex] \dfrac{30a^7b^4}{2a^4b^2} & \text{Now use the quotient rule to subtract exponents} \\[2ex] 15a^3b^2 & \text{Our Solution} \end{array}$$

Example 17.

$$\begin{array}{rl} \dfrac{3m^8n^{12}}{(m^2n^3)^3} & \text{Use power rule in denominator} \\[2ex] \dfrac{3m^8n^{12}}{m^6n^9} & \text{Use quotient rule} \\[2ex] 3m^2n^3 & \text{Our solution} \end{array}$$

Example 18.

$$\begin{array}{rl} \left(\dfrac{3\text{ab}^2(2a^4b^2)^3}{6a^5b^7}\right)^2 & \text{Simplify inside parenthesis first, using power rule in numerator} \\[2ex] \left(\dfrac{3\text{ab}^2(8a^{12}b^6)}{6a^5b^7}\right)^2 & \text{Simplify numerator using product rule} \\[2ex] \left(\dfrac{24a^{13}b^8}{6a^5b^7}\right)^2 & \text{Simplify using the quotient rule} \\[2ex] (4a^8b)^2 & \text{Now that the parenthesis are simplified, use the power rules} \\ 16a^{16}b^2 & \text{Our Solution} \end{array}$$

Clearly these problems can quickly become quite involved. Remember to follow order of operations as a guide, simplify inside parenthesis first, then power rules, then product and quotient rules.

2.1 Practice - Exponent Properties

Simplify.

1) $4 \cdot 4^4 \cdot 4^4$

3) $4 \cdot 2^2$

5) $3m \cdot 4\text{mn}$

7) $2m^4n^2 \cdot 4nm^2$

9) $(3^3)^4$

11) $(4^4)^2$

13) $(2u^3v^2)^2$

15) $(2a^4)^4$

17) $\frac{4^5}{4^3}$

19) $\frac{3^2}{3}$

21) $\frac{3nm^2}{3n}$

23) $\frac{4x^3y^4}{3xy^3}$

25) $(x^3y^4 \cdot 2x^2y^3)^2$

27) $2x(x^4y^4)^4$

29) $\frac{2x^7y^5}{3x^3y \cdot 4x^2y^3}$

31) $\left(\frac{(2x)^3}{x^3}\right)^2$

33) $\left(\frac{2y^{17}}{(2x^2y^4)^4}\right)^3$

35) $\left(\frac{2mn^4 \cdot 2m^4n^4}{mn^4}\right)^3$

37) $\frac{2xy^5 \cdot 2x^2y^3}{2xy^4 \cdot y^3}$

39) $\frac{q^3r^2 \cdot (2p^2q^2r^3)^2}{2p^3}$

41) $\left(\frac{zy^3 \cdot z^3x^4y^4}{x^3y^3z^3}\right)^4$

43) $\frac{2x^2y^2z^6 \cdot 2zx^2y^2}{(x^2z^3)^2}$

2) $4 \cdot 4^4 \cdot 4^2$

4) $3 \cdot 3^3 \cdot 3^2$

6) $3x \cdot 4x^2$

8) $x^2y^4 \cdot xy^2$

10) $(4^3)^4$

12) $(3^2)^3$

14) $(xy)^3$

16) $(2xy)^4$

18) $\frac{3^7}{3^3}$

20) $\frac{3^4}{3}$

22) $\frac{x^2y^4}{4xy}$

24) $\frac{xy^3}{4xy}$

26) $(u^2v^2 \cdot 2u^4)^3$

28) $\frac{3vu^5 \cdot 2v^3}{uv^2 \cdot 2u^3v}$

30) $\frac{2ba^7 \cdot 2b^4}{ba^2 \cdot 3a^3b^4}$

32) $\frac{2a^2b^2a^7}{(ba^4)^2}$

34) $\frac{yx^2 \cdot (y^4)^2}{2y^4}$

36) $\frac{n^3(n^4)^2}{2mn}$

38) $\frac{(2y^3x^2)^2}{2x^2y^4 \cdot x^2}$

40) $\frac{2x^4y^5 \cdot 2z^{10}x^2y^7}{(xy^2z^2)^4}$

42) $\left(\frac{2q^3p^3r^4 \cdot 2p^3}{(qrp^3)^2}\right)^4$

Answers to Exponent Properties

1) 4^9

2) 4^7

3) 2^4

4) 3^6

5) $12m^2n$

6) $12x^3$

7) $8m^6n^3$

8) x^3y^6

9) 3^{12}

10) 4^{12}

11) 4^8

12) 3^6

13) $4u^6v^4$

14) x^3y^3

15) $16a^{16}$

16) $16x^4y^4$

17) 4^2

18) 3^4

19) 3

20) 3^3

21) m^2

22) $\frac{xy^3}{4}$

23) $\frac{4x^2y}{3}$

24) $\frac{y^2}{4}$

25) $4x^{10}y^{14}$

26) $8u^{18}v^6$

27) $2x^{17}y^{16}$

28) $3uv$

29) $\frac{x^2y}{6}$

30) $\frac{4a^2}{3}$

31) 64

32) $2a$

33) $\frac{y^3}{512x^{24}}$

34) $\frac{y^5x^2}{2}$

35) $64m^{12}n^{12}$

36) $\frac{n^{10}}{2m}$

37) $2x^2y$

38) $2y^2$

39) $2q^7r^8p$

40) $4x^2y^4z^2$

41) $x^4y^{16}z^4$

42) $256q^4r^8$

43) $4y^4z$

2.2 Negative Exponents

Objective: Simplify expressions with negative exponents using the properties of exponents.

There are a few special exponent properties that deal with exponents that are not positive. The first is considered in the following example, which is worded out 2 different ways:

Example 1.

$$\frac{a^3}{a^3} \quad \text{Use the quotient rule to subtract exponents}$$

$$a^0 \quad \text{Our Solution, but now we consider the problem } a \text{ the second way:}$$

$$\frac{a^3}{a^3} \quad \text{Rewrite exponents as repeated multiplication}$$

$$\frac{aaa}{aaa} \quad \text{Reduce out all the } a's$$

$$\frac{1}{1} = 1 \quad \text{Our Solution, when we combine the two solutions we get:}$$

$$a^0 = 1 \quad \text{Our final result.}$$

This final result is an imporant property known as the zero power rule of exponents

Zero Power Rule of Exponents: $a^0 = 1$

Any number or expression raised to the zero power will always be 1. This is illustrated in the following example.

Example 2.

$$(3x^2)^0 \quad \text{Zero power rule}$$

$$1 \quad \text{Our Solution}$$

Another property we will consider here deals with negative exponents. Again we will solve the following example two ways.

Example 3.

$$\frac{a^3}{a^5} \quad \text{Using the quotient rule, subtract exponents}$$

$$a^{-2} \quad \text{Our Solution, but we will also solve this problem another way.}$$

$$\frac{a^3}{a^5} \quad \text{Rewrite exponents as repeated multiplication}$$

$$\frac{aaa}{aaaaa} \quad \text{Reduce three } a's \text{ out of top and bottom}$$

$$\frac{1}{aa} \quad \text{Simplify to exponents}$$

$$\frac{1}{a^2} \quad \text{Our Solution, putting these solutions together gives:}$$

$$a^{-2} = \frac{1}{a^2} \quad \text{Our Final Solution}$$

This example illustrates an important property of exponents. Negative exponents yield the reciprocal of the base. Once we take the reciprical the exponent is now positive. Also, it is important to note a negative exponent does not mean the expression is negative, only that we need the reciprocal of the base. Following are the rules of negative exponents

$$\boldsymbol{a^{-m} = \frac{1}{m}}$$

Rules of Negative Exponets: $\boldsymbol{\frac{1}{a^{-m}} = a^m}$

$$\boldsymbol{\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}}$$

Negative exponents can be combined in several different ways. As a general rule if we think of our expression as a fraction, negative exponents in the numerator must be moved to the denominator, likewise, negative exponents in the denominator need to be moved to the numerator. When the base with exponent moves, the exponent is now positive. This is illustrated in the following example.

Example 4.

$$\frac{a^3b^{-2}c}{2d^{-1}e^{-4}f^2} \quad \text{Negative exponents on } b, d, \text{ and } e \text{ need to flip}$$

$$\frac{a^3cde^4}{2b^2f^2} \quad \text{Our Solution}$$

As we simplified our fraction we took special care to move the bases that had a negative exponent, but the expression itself did not become negative because of those exponents. Also, it is important to remember that exponents only effect what they are attached to. The 2 in the denominator of the above example does not have an exponent on it, so it does not move with the d.

We now have the following nine properties of exponents. It is important that we are very familiar with all of them.

Properties of Exponents

$\boldsymbol{a^m a^n = a^{m+n}}$	$\boldsymbol{(ab)^m = a^m b^m}$	$\boldsymbol{a^{-m} = \frac{1}{a^m}}$
$\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$	$\boldsymbol{\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}}$	$\boldsymbol{\frac{1}{a^{-m}} = a^m}$
$\boldsymbol{(a^m)^n = a^{mn}}$	$\boldsymbol{a^0 = 1}$	$\boldsymbol{\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}}$

World View Note: Nicolas Chuquet, the French mathematician of the 15th century wrote $12^{1\bar{m}}$ to indicate $12x^{-1}$. This was the first known use of the negative exponent.

Simplifying with negative exponents is much the same as simplifying with positive exponents. It is the advice of the author to keep the negative exponents until the end of the problem and then move them around to their correct location (numerator or denominator). As we do this it is important to be very careful of rules for adding, subtracting, and multiplying with negatives. This is illustrated in the following examples

Example 5.

$$\frac{4x^{-5}y^{-3}\cdot 3x^3y^{-2}}{6x^{-5}y^3}$$ Simplify numerator with product rule, adding exponents

$$\frac{12x^{-2}y^{-5}}{6x^{-5}y^3}$$ Quotient rule to subtract exponets, be careful with negatives!

$(-2)-(-5)=(-2)+5=3$

$(-5)-3=(-5)+(-3)=-8$

$$2x^3y^{-8}$$ Negative exponent needs to move down to denominator

$$\frac{2x^3}{y^8}$$ Our Solution

Example 6.

$$\frac{(3\mathrm{ab}^3)^{-2}\mathrm{ab}^{-3}}{2a^{-4}b^0}$$ In numerator, use power rule with -2, multiplying exponents

In denominator, $b^0=1$

$$\frac{3^{-2}a^{-2}b^{-6}\mathrm{ab}^{-3}}{2a^{-4}}$$ In numerator, use product rule to add exponents

$\frac{3^{-2}a^{-1}b^{-9}}{2a^{-4}}$ Use quotient rule to subtract exponents, be careful with negatives

$(-1)-(-4)=(-1)+4=3$

$\frac{3^{-2}a^{3}b^{-9}}{2}$ Move 3 and b to denominator because of negative exponents

$\frac{a^3}{3^2 2b^9}$ Evaluate $3^2 2$

$\frac{a^3}{18b^9}$ Our Solution

In the previous example it is important to point out that when we simplified 3^{-2} we moved the three to the denominator and the exponent became positive. We did not make the number negative! Negative exponents never make the bases negative, they simply mean we have to take the reciprocal of the base. One final example with negative exponents is given here.

Example 7.

$\left(\frac{3x^{-2}y^5z^3\cdot 6x^{-6}y^{-2}z^{-3}}{9(x^2y^{-2})^{-3}}\right)^{-3}$ In numerator, use product rule, adding exponents
In denominator, use power rule, multiplying exponets

$\left(\frac{18x^{-8}y^3z^0}{9x^{-6}y^6}\right)^{-3}$ Use quotient rule to subtract exponents,

be careful with negatives:

$(-8)-(-6)=(-8)+6=-2$

$3-6=3+(-6)=-3$

$(2x^{-2}y^{-3}z^0)^{-3}$ Parenthesis are done, use power rule with -3

$2^{-3}x^6y^9z^0$ Move 2 with negative exponent down and $z^0=1$

$\frac{x^6y^9}{2^3}$ Evaluate 2^3

$\frac{x^6y^9}{8}$ Our Solution

2.2 Practice - Negative Exponents

Simplify. Your answer should contain only positive expontents.

1) $2x^4y^{-2}\cdot(2xy^3)^4$

2) $2a^{-2}b^{-3}\cdot(2a^0b^4)^4$

3) $(a^4b^{-3})^3\cdot 2a^3b^{-2}$

4) $2x^3y^2\cdot(2x^3)^0$

5) $(2x^2y^2)^4x^{-4}$

6) $(m^0n^3\cdot 2m^{-3}n^{-3})^0$

7) $(x^3y^4)^3\cdot x^{-4}y^4$

8) $2m^{-1}n^{-3}\cdot(2m^{-1}n^{-3})^4$

9) $\frac{2x^{-3}y^2}{3x^{-3}y^3\cdot 3x^0}$

10) $\frac{3y^3}{3yx^3\cdot 2x^4y^{-3}}$

11) $\frac{4xy^{-3}\cdot x^{-4}y^0}{4y^{-1}}$

12) $\frac{3x^3y^2}{4y^{-2}\cdot 3x^{-2}y^{-4}}$

13) $\frac{u^2v^{-1}}{2u^0v^4\cdot 2uv}$

14) $\frac{2xy^2\cdot 4x^3y^{-4}}{4x^{-4}y^{-4}\cdot 4x}$

15) $\frac{u^2}{4u^0v^3\cdot 3v^2}$

16) $\frac{2x^{-2}y^2}{4yx^2}$

17) $\frac{2y}{(x^0y^2)^4}$

18) $\frac{(a^4)^4}{2b}$

19) $(\frac{2a^2b^3}{a^{-1}})^4$

20) $(\frac{2y^{-4}}{x^2})^{-2}$

21) $\frac{2nm^4}{(2m^2n^2)^4}$

22) $\frac{2y^2}{(x^4y^0)^{-4}}$

23) $\frac{(2mn)^4}{m^0n^{-2}}$

24) $\frac{2x^{-3}}{(x^4y^{-3})^{-1}}$

25) $\frac{y^3\cdot x^{-3}y^2}{(x^4y^2)^3}$

26) $\frac{2x^{-2}y^0\cdot 2xy^4}{(xy^0)^{-1}}$

27) $\frac{2u^{-2}v^3\cdot(2uv^4)^{-1}}{2u^{-4}v^0}$

28) $\frac{2yx^2\cdot x^{-2}}{(2x^0y^4)^{-1}}$

29) $(\frac{2x^0\cdot y^4}{y^4})^3$

30) $\frac{u^{-3}v^{-4}}{2v(2u^{-3}v^4)^0}$

31) $\frac{y(2x^4y^2)^2}{2x^4y^0}$

32) $\frac{b^{-1}}{(2a^4b^0)^0\cdot 2a^{-3}b^2}$

33) $\frac{2yzx^2}{2x^4y^4z^{-2}\cdot(zy^2)^4}$

34) $\frac{2b^4c^{-2}\cdot(2b^3c^2)^{-4}}{a^{-2}b^4}$

35) $\frac{2kh^0\cdot 2h^{-3}k^0}{(2kj^3)^2}$

36) $(\frac{(2x^{-3}y^0z^{-1})^3\cdot x^{-3}y^2}{2x^3})^{-2}$

37) $\frac{(cb^3)^2\cdot 2a^{-3}b^2}{(a^3b^{-2}c^3)^3}$

38) $\frac{2q^4\cdot m^2p^2q^4}{(2m^{-4}p^2)^3}$

39) $\frac{(yx^{-4}z^2)^{-1}}{z^3\cdot x^2y^3z^{-1}}$

40) $\frac{2mpn^{-3}}{(m^0n^{-4}p^2)^3\cdot 2n^2p^0}$

2.2

Answers to Negative Exponents

1) $32x^8y^{10}$

2) $\frac{32b^{13}}{a^2}$

3) $\frac{2a^{15}}{b^{11}}$

4) $2x^3y^2$

5) $16x^4y^8$

6) 1

7) $y^{16}x^5$

8) $\frac{32}{m^5n^{15}}$

9) $\frac{2}{9y}$

10) $\frac{y^5}{2x^7}$

11) $\frac{1}{y^2x^3}$

12) $\frac{y^8x^5}{4}$

13) $\frac{u}{4v^6}$

14) $\frac{x^7y^2}{2}$

15) $\frac{u^2}{12v^5}$

16) $\frac{y}{2x^4}$

17) $\frac{2}{y^7}$

18) $\frac{a^{16}}{2b}$

19) $16a^{12}b^{12}$

20) $\frac{y^8x^4}{4}$

21) $\frac{1}{8m^4n^7}$

22) $2x^{16}y^2$

23) $16n^6m^4$

24) $\frac{2x}{y^3}$

25) $\frac{1}{x^{15}y}$

26) $4y^4$

27) $\frac{u}{2v}$

28) $4y^5$

29) 8

30) $\frac{1}{2u^3v^5}$

31) $2y^5x^4$

32) $\frac{a^3}{2b^3}$

33) $\frac{1}{x^2y^{11}z}$

34) $\frac{a^2}{8c^{10}b^{12}}$

35) $\frac{1}{h^3kj^6}$

36) $\frac{x^{30}z^6}{16y^4}$

37) $\frac{2b^{14}}{a^{12}c^7}$

38) $\frac{m^{14}q^8}{4p^4}$

39) $\frac{x^2}{y^4z^4}$

40) $\frac{mn^7}{p^5}$

2.3 Scientific Notation

Objective: Multiply and divide expressions using scientific notation and exponent properties.

One application of exponent properties comes from scientific notation. Scientific notation is used to represent really large or really small numbers. An example of really large numbers would be the distance that light travels in a year in miles. An example of really small numbers would be the mass of a single hydrogen atom in grams. Doing basic operations such as multiplication and division with these numbers would normally be very combersome. However, our exponent properties make this process much simpler.

First we will take a look at what scientific notation is. Scientific notation has two parts, a number between one and ten (it can be equal to one, but not ten), and that number multiplied by ten to some exponent.

$$\textbf{Scientific Notation: } \boldsymbol{a \times 10^b} \textbf{ where } \boldsymbol{1 \leqslant a < 10}$$

The exponent, b, is very important to how we convert between scientific notation and normal numbers, or standard notation. The exponent tells us how many times we will multiply by 10. Multiplying by 10 in affect moves the decimal point one place. So the exponent will tell us how many times the exponent moves between scientific notation and standard notation. To decide which direction to move the decimal (left or right) we simply need to remember that positive exponents mean in standard notation we have a big number (bigger than ten) and negative exponents mean in standard notation we have a small number (less than one).

Keeping this in mind, we can easily make conversions between standard notation and scientific notation.

Example 1.

Convert $14,200$ to scientific notation	Put decimal after first nonzero number
1.42	Exponent is how many times decimal moved, 4
$\times 10^4$	Positive exponent, standard notation is big
1.42×10^4	Our Solution

Example 2.

Convert 0.0042 to scientific notation	Put decimal after first nonzero number
4.2	Exponent is how many times decimal moved, 3
$\times 10^{-3}$	Negative exponent, standard notation is small
4.2×10^{-3}	Our Solution

Example 3.

Convert 3.21×10^5 to standard notation	Positive exponent means standard notation big number. Move decimal right 5 places
$321,000$	Our Solution

Example 4.

Conver 7.4×10^{-3} to standard notation	Negative exponent means standard notation is a small number. Move decimal left 3 places
0.0074	Our Solution

Converting between standard notation and scientific notation is important to understand how scientific notation works and what it does. Here our main interest is to be able to multiply and divide numbers in scientific notation using exponent properties. The way we do this is first do the operation with the front number (multiply or divide) then use exponent properties to simplify the 10's. Scientific notation is the only time where it will be allowed to have negative exponents in our final solution. The negative exponent simply informs us that we are dealing with small numbers. Consider the following examples.

Example 5.

$(2.1\times10^{-7})(3.7\times10^5)$	Deal with numbers and $10's$ separately
$(2.1)(3.7)=7.77$	Multiply numbers
$10^{-7}10^5=10^{-2}$	Use product rule on $10's$ and add exponents
7.77×10^{-2}	Our Solution

Example 6.

$\dfrac{4.96\times10^4}{3.1\times10^{-3}}$	Deal with numbers and $10's$ separately
$\dfrac{4.96}{3.1}=1.6$	Divide Numbers
$\dfrac{10^4}{10^{-3}}=10^7$	Use quotient rule to subtract exponents, be careful with negatives!
	Be careful with negatives, $4-(-3)=4+3=7$
1.6×10^7	Our Solution

Example 7.

$(1.8\times10^{-4})^3$	Use power rule to deal with numbers and $10's$ separately
$1.8^3=5.832$	Evaluate 1.8^3
$(10^{-4})^3=10^{-12}$	Multiply exponents
5.832×10^{-12}	Our Solution

Often when we multiply or divide in scientific notation the end result is not in scientific notation. We will then have to convert the front number into scientific notation and then combine the 10's using the product property of exponents and adding the exponents. This is shown in the following examples.

Example 8.

$(4.7\times10^{-3})(6.1\times10^{9})$	Deal with numbers and $10's$ separately
$(4.7)(6.1)=28.67$	Multiply numbers
2.867×10^{1}	Convert this number into scientific notation
$\mathbf{10^1}10^{-3}10^{9}=10^{7}$	Use product rule, add exponents, using 10^1 from conversion
2.867×10^{7}	Our Solution

World View Note: Archimedes (287 BC - 212 BC), the Greek mathematician, developed a system for representing large numbers using a system very similar to scientific notation. He used his system to calculate the number of grains of sand it would take to fill the universe. His conclusion was 10^{63} grains of sand because he figured the universe to have a diameter of 10^{14} stadia or about 2 light years.

Example 9.

$\dfrac{2.014\times10^{-3}}{3.8\times10^{-7}}$	Deal with numbers and $10's$ separately
$\dfrac{2.014}{3.8}=0.53$	Divide numbers
$0.53=5.3\times10^{-1}$	Change this number into scientific notation
$\dfrac{\mathbf{10^{-1}}10^{-3}}{10^{-7}}=10^{3}$	Use product and quotient rule, using 10^{-1} from the conversion
	Be careful with signs:
	$(-1)+(-3)-(-7)=(-1)+(-3)+7=3$
5.3×10^{3}	Our Solution

2.3 Practice - Scientific Notation

Write each number in scientific notiation

1) 885
2) 0.000744
3) 0.081
4) 1.09
5) 0.039
6) 15000

Write each number in standard notation

7) $8.7 \text{ x } 10^5$
8) $2.56 \text{ x } 10^2$
9) $9 \text{ x } 10^{-4}$
10) $5 \text{ x } 10^4$
11) $2 \text{ x } 10^0$
12) $6 \text{ x } 10^{-5}$

Simplify. Write each answer in scientific notation.

13) $(7 \text{ x } 10^{-1})(2 \text{ x } 10^{-3})$
14) $(2 \times 10^{-6})(8.8 \times 10^{-5})$
15) $(5.26 \text{ x } 10^{-5})(3.16 \text{ x } 10^{-2})$
16) $(5.1 \times 10^6)(9.84 \times 10^{-1})$
17) $(2.6 \text{ x } 10^{-2})(6 \text{ x } 10^{-2})$
18) $\frac{7.4 \times 10^4}{1.7 \times 10^{-4}}$
19) $\frac{4.9 \times 10^1}{2.7 \times 10^{-3}}$
20) $\frac{7.2 \times 10^{-1}}{7.32 \times 10^{-1}}$
21) $\frac{5.33 \times 10^{-6}}{9.62 \times 10^{-2}}$
22) $\frac{3.2 \times 10^{-3}}{5.02 \times 10^0}$
23) $(5.5 \times 10^{-5})^2$
24) $(9.6 \times 10^3)^{-4}$
25) $(7.8 \times 10^{-2})^5$
26) $(5.4 \times 10^6)^{-3}$
27) $(8.03 \times 10^4)^{-4}$
28) $(6.88 \times 10^{-4})(4.23 \times 10^1)$
29) $\frac{6.1 \times 10^{-6}}{5.1 \times 10^{-4}}$
30) $\frac{8.4 \times 10^5}{7 \times 10^{-2}}$
31) $(3.6 \times 10^0)(6.1 \times 10^{-3})$
32) $(3.15 \times 10^3)(8 \times 10^{-1})$
33) $(1.8 \times 10^{-5})^{-3}$
34) $\frac{9.58 \times 10^{-2}}{1.14 \times 10^{-3}}$
35) $\frac{9 \times 10^4}{7.83 \times 10^{-2}}$
36) $(8.3 \times 10^1)^5$
37) $\frac{3.22 \times 10^{-3}}{7 \times 10^{-6}}$
38) $\frac{5 \times 10^6}{6.69 \times 10^2}$
39) $\frac{2.4 \times 10^{-6}}{6.5 \times 10^0}$
40) $(9 \times 10^{-2})^{-3}$
41) $\frac{6 \times 10^3}{5.8 \times 10^{-3}}$
42) $(2 \times 10^4)(6 \times 10^1)$

Answers to Scientific Notation

1) 8.85×10^{2}
2) 7.44×10^{-4}
3) 8.1×10^{-2}
4) 1.09×10^{0}
5) 3.9×10^{-2}
6) 1.5×10^{4}
7) 870000
8) 256
9) 0.0009
10) 50000
11) 2
12) 0.00006
13) 1.4×10^{-3}
14) 1.76×10^{-10}
15) 1.662×10^{-6}
16) 5.018×10^{6}
17) 1.56×10^{-3}
18) 4.353×10^{8}
19) 1.815×10^{4}
20) 9.836×10^{-1}
21) 5.541×10^{-5}
22) 6.375×10^{-4}
23) 3.025×10^{-9}
24) 1.177×10^{-16}
25) 2.887×10^{-6}
26) 6.351×10^{-21}
27) 2.405×10^{-20}
28) 2.91×10^{-2}
29) 1.196×10^{-2}
30) 1.2×10^{7}
31) 2.196×10^{-2}
32) 2.52×10^{3}
33) 1.715×10^{14}
34) 8.404×10^{1}
35) 1.149×10^{6}
36) 3.939×10^{9}
37) 4.6×10^{2}
38) 7.474×10^{3}
39) 3.692×10^{-7}
40) 1.372×10^{3}
41) 1.034×10^{6}
42) 1.2×10^{6}

2.4 Introduction to Polynomials

Objective: Evaluate, add, and subtract polynomials.

Many applications in mathematics have to do with what are called polynomials. Polynomials are made up of terms. **Terms** are a product of numbers and/or variables. For example, $5x$, $2y^2$, -5, ab^3c, and x are all terms. Terms are connected to each other by addition or subtraction. Expressions are often named based on the number of terms in them. A **monomial** has one term, such as $3x^2$. A **binomial** has two terms, such as $a^2 - b^2$. A Trinomial has three terms, such as $ax^2 + bx + c$. The term **polynomial** means many terms. Monomials, binomials, trinomials, and expressions with more terms all fall under the umbrella of "polynomials".

If we know what the variable in a polynomial represents we can replace the variable with the number and evaluate the polynomial as shown in the following example.

Example 1.

$$
\begin{array}{rl}
2x^2-4x+6 \text{ when } x=-4 & \text{Replace variable } x \text{ with } -4\\
2(-4)^2-4(-4)+6 & \text{Exponents first}\\
2(16)-4(-4)+6 & \text{Multiplication (we can do all terms at once)}\\
32+16+6 & \text{Add}\\
54 & \text{Our Solution}
\end{array}
$$

It is important to be careful with negative variables and exponents. Remember the exponent only effects the number it is physically attached to. This means $-3^2=-9$ because the exponent is only attached to the 3. Also, $(-3)^2=9$ because the exponent is attached to the parenthesis and effects everything inside. When we replace a variable with parenthesis like in the previous example, the substituted value is in parenthesis. So the $(-4)^2=16$ in the example. However, consider the next example.

Example 2.

$$
\begin{array}{rl}
-x^2+2x+6 \text{ when } x=3 & \text{Replace variable } x \text{ with } 3\\
-(3)^2+2(3)+6 & \text{Exponent only on the 3, not negative}\\
-9+2(3)+6 & \text{Multiply}\\
-9+6+6 & \text{Add}\\
3 & \text{Our Solution}
\end{array}
$$

World View Note: Ada Lovelace in 1842 described a Difference Engine that would be used to caluclate values of polynomials. Her work became the foundation for what would become the modern computer (the programming language Ada was named in her honor), more than 100 years after her death from cancer.

Generally when working with polynomials we do not know the value of the variable, so we will try and simplify instead. The simplest operation with polynomials is addition. When adding polynomials we are mearly combining like terms. Consider the following example

Example 3.

$$\begin{array}{rl}(4x^3-2x+8)+(3x^3-9x^2-11) & \text{Combine like terms } 4x^3+3x^3 \text{ and } 8-11\\ 7x^3-9x^2-2x-3 & \text{Our Solution}\end{array}$$

Generally final answers for polynomials are written so the exponent on the variable counts down. Example 3 demonstrates this with the exponent counting down 3, 2, 1, 0 (recall $x^0=1$). Subtracting polynomials is almost as fast. One extra step comes from the minus in front of the parenthesis. When we have a negative in front of parenthesis we distribute it through, changing the signs of everything inside. The same is done for the subtraction sign.

Example 4.

$$\begin{array}{rl}(5x^2-2x+7)-(3x^2+6x-4) & \text{Distribute negative through second part}\\ 5x^2-2x+7-3x^2-6x+4 & \text{Combine like terms } 5x^2-3x^3, -2x-6x, \text{ and } 7+4\\ 2x^2-8x+11 & \text{Our Solution}\end{array}$$

Addition and subtraction can also be combined into the same problem as shown in this final example.

Example 5.

$$\begin{array}{rl}(2x^2-4x+3)+(5x^2-6x+1)-(x^2-9x+8) & \text{Distribute negative through}\\ 2x^2-4x+3+5x^2-6x+1-x^2+9x-8 & \text{Combine like terms}\\ 6x^2-x-4 & \text{Our Solution}\end{array}$$

2.4 Practice - Introduction to Polynomials

Simplify each expression.

1) $-a^3 - a^2 + 6a - 21$ when $a = -4$

2) $n^2 + 3n - 11$ when $n = -6$

3) $n^3 - 7n^2 + 15n - 20$ when $n = 2$

4) $n^3 - 9n^2 + 23n - 21$ when $n = 5$

5) $-5n^4 - 11n^3 - 9n^2 - n - 5$ when $n = -1$

6) $x^4 - 5x^3 - x + 13$ when $x = 5$

7) $x^2 + 9x + 23$ when $x = -3$

8) $-6x^3 + 41x^2 - 32x + 11$ when $x = 6$

9) $x^4 - 6x^3 + x^2 - 24$ when $x = 6$

10) $m^4 + 8m^3 + 14m^2 + 13m + 5$ when $m = -6$

11) $(5p - 5p^4) - (8p - 8p^4)$

12) $(7m^2 + 5m^3) - (6m^3 - 5m^2)$

13) $(3n^2 + n^3) - (2n^3 - 7n^2)$

14) $(x^2 + 5x^3) + (7x^2 + 3x^3)$

15) $(8n + n^4) - (3n - 4n^4)$

16) $(3v^4 + 1) + (5 - v^4)$

17) $(1 + 5p^3) - (1 - 8p^3)$

18) $(6x^3 + 5x) - (8x + 6x^3)$

19) $(5n^4 + 6n^3) + (8 - 3n^3 - 5n^4)$

20) $(8x^2 + 1) - (6 - x^2 - x^4)$

21) $(3 + b^4) + (7 + 2b + b^4)$

22) $(1 + 6r^2) + (6r^2 - 2 - 3r^4)$

23) $(8x^3+1)-(5x^4-6x^3+2)$

24) $(4n^4+6)-(4n-1-n^4)$

25) $(2a+2a^4)-(3a^2-5a^4+4a)$

26) $(6v+8v^3)+(3+4v^3-3v)$

27) $(4p^2-3-2p)-(3p^2-6p+3)$

28) $(7+4m+8m^4)-(5m^4+1+6m)$

29) $(4b^3+7b^2-3)+(8+5b^2+b^3)$

30) $(7n+1-8n^4)-(3n+7n^4+7)$

31) $(3+2n^2+4n^4)+(n^3-7n^2-4n^4)$

32) $(7x^2+2x^4+7x^3)+(6x^3-8x^4-7x^2)$

33) $(n-5n^4+7)+(n^2-7n^4-n)$

34) $(8x^2+2x^4+7x^3)+(7x^4-7x^3+2x^2)$

35) $(8r^4-5r^3+5r^2)+(2r^2+2r^3-7r^4+1)$

36) $(4x^3+x-7x^2)+(x^2-8+2x+6x^3)$

37) $(2n^2+7n^4-2)+(2+2n^3+4n^2+2n^4)$

38) $(7b^3-4b+4b^4)-(8b^3-4b^2+2b^4-8b)$

39) $(8-b+7b^3)-(3b^4+7b-8+7b^2)+(3-3b+6b^3)$

40) $(1-3n^4-8n^3)+(7n^4+2-6n^2+3n^3)+(4n^3+8n^4+7)$

41) $(8x^4+2x^3+2x)+(2x+2-2x^3-x^4)-(x^3+5x^4+8x)$

42) $(6x-5x^4-4x^2)-(2x-7x^2-4x^4-8)-(8-6x^2-4x^4)$

Answers to Introduction to Polynomials

1) 3

2) 7

3) -10

4) -6

5) -7

6) 8

7) 5

8) -1

9) 12

10) -1

11) $3p^4-3p$

12) $-m^3+12m^2$

13) $-n^3+10n^2$

14) $8x^3+8x^2$

15) $5n^4+5n$

16) $2v^4+6$

17) $13p^3$

18) $-3x$

19) $3n^3+8$

20) x^4+9x^2-5

21) $2b^4+2b+10$

22) $-3r^4+12r^2-1$

23) $-5x^4+14x^3-1$

24) $5n^4-4n+7$

25) $7a^4-3a^2-2a$

26) $12v^3+3v+3$

27) p^2+4p-6

28) $3m^4-2m+6$

29) $5b^3+12b^2+5$

30) $-15n^4+4n-6$

31) n^3-5n^2+3

32) $-6x^4+13x^3$

33) $-12n^4+n^2+7$

34) $9x^2+10x^2$

35) $r^4-3r^3+7r^2+1$

36) $10x^3-6x^2+3x-8$

37) $9n^4+2n^3+6n^2$

38) $2b^4-b^3+4b^2+4b$

39) $-3b^4+13b^3-7b^2-11b+19$

40) $12n^4-n^3-6n^2+10$

41) $2x^4-x^3-4x+2$

42) $3x^4+9x^2+4x$

2.5 Multiplying Polynomials

Objective: Multiply polynomials.

Multiplying polynomials can take several different forms based on what we are multiplying. We will first look at multiplying monomials, then monomials by polynomials and finish with polynomials by polynomials.

Multiplying monomials is done by multiplying the numbers or coefficients and then adding the exponents on like factors. This is shown in the next example.

Example 1.

$$\begin{array}{rl} (4x^3y^4z)(2x^2y^6z^3) & \text{Multiply numbers and add exponents for } x, y, \text{ and } z \\ 8x^5y^{10}z^4 & \text{Our Solution} \end{array}$$

In the previous example it is important to remember that the z has an exponent of 1 when no exponent is written. Thus for our answer the z has an exponent of $1+3=4$. Be very careful with exponents in polynomials. If we are adding or subtracting the exponnets will stay the same, but when we multiply (or divide) the exponents will be changing.

Next we consider multiplying a monomial by a polynomial. We have seen this operation before with distributing through parenthesis. Here we will see the exact same process.

Example 2.

$$\begin{array}{rl} 4x^3(5x^2-2x+5) & \text{Distribute the } 4x^3\text{, multiplying numbers, adding exponents} \\ 20x^5-8x^4+20x^3 & \text{Our Solution} \end{array}$$

Following is another example with more variables. When distributing the exponents on a are added and the exponents on b are added.

Example 3.

$$\begin{array}{rl} 2a^3b(3\mathrm{ab}^2-4a) & \text{Distribute, multiplying numbers and adding exponents} \\ 6a^4b^3-8a^4b & \text{Our Solution} \end{array}$$

There are several different methods for multiplying polynomials. All of which work, often students prefer the method they are first taught. Here three methods will be discussed. All three methods will be used to solve the same two multiplication problems.

Multiply by Distributing

Just as we distribute a monomial through parenthesis we can distribute an entire polynomial. As we do this we take each term of the second polynomial and put it in front of the first polynomial.

Example 4.

$$\begin{array}{rl}
(4x+7y)(3x-2y) & \text{Distribute } (4x+7y) \text{ through parenthesis} \\
3x(\boldsymbol{4x+7y})-2y(\boldsymbol{4x+7y}) & \text{Distribute the } 3x \text{ and } -2y \\
12x^2+21xy-8xy-14y^2 & \text{Combine like terms } 21xy-8xy \\
12x^2+13xy-14y^2 & \text{Our Solution}
\end{array}$$

This example illustrates an important point, the negative/subtraction sign stays with the $2y$. Which means on the second step the negative is also distributed through the last set of parenthesis.

Multiplying by distributing can easily be extended to problems with more terms. First distribute the front parenthesis onto each term, then distribute again!

Example 5.

$$\begin{array}{rl}
(2x-5)(4x^2-7x+3) & \text{Distribute } (2x-5) \text{ through parenthesis} \\
4x^2(\boldsymbol{2x-5})-7x(\boldsymbol{2x-5})+3(\boldsymbol{2x-5}) & \text{Distribute again through each parenthesis} \\
8x^3-20x^2-14x^2+35x+6x-15 & \text{Combine like terms} \\
8x^3-34x^2+41x-15 & \text{Our Solution}
\end{array}$$

This process of multiplying by distributing can easily be reversed to do an important procedure known as factoring. Factoring will be addressed in a future lesson.

Multiply by FOIL

Another form of multiplying is known as FOIL. Using the FOIL method we multiply each term in the first binomial by each term in the second binomial. The letters of FOIL help us remember every combination. F stands for First, we multiply the first term of each binomial. O stand for Outside, we multiply the outside two terms. I stands for Inside, we multiply the inside two terms. L stands for Last, we multiply the last term of each binomial. This is shown in the next example:

Example 6.

$$\begin{array}{rl}
(4x+7y)(3x-2y) & \text{Use FOIL to multiply} \\
(4x)(3x)=12x^2 & F-\text{First terms } (4x)(3x) \\
(4x)(-2y)=-8xy & O-\text{Outside terms } (4x)(-2y) \\
(7y)(3x)=21xy & I-\text{Inside terms } (7y)(3x) \\
(7y)(-2y)=-14y^2 & L-\text{Last terms } (7y)(-2y) \\
12x^2-8xy+21xy-14y^2 & \text{Combine like terms } -8xy+21xy \\
12x^2+13xy-14y^2 & \text{Our Solution}
\end{array}$$

Some students like to think of the FOIL method as distributing the first term $4x$ through the $(3x - 2y)$ and distributing the second term $7y$ through the $(3x - 2y)$. Thinking about FOIL in this way makes it possible to extend this method to problems with more terms.

Example 7.

$$\begin{array}{rl}
(2x-5)(4x^2-7x+3) & \text{Distribute } 2x \text{ and } -5 \\
(2x)(4x^2)+(2x)(-7x)+(2x)(3)-5(4x^2)-5(-7x)-5(3) & \text{Multiply out each term} \\
8x^3-14x^2+6x-20x^2+35x-15 & \text{Combine like terms} \\
8x^3-34x^2+41x-15 & \text{Our Solution}
\end{array}$$

The second step of the FOIL method is often not written, for example, consider the previous example, a student will often go from the problem $(4x + 7y)(3x - 2y)$ and do the multiplication mentally to come up with $12x^2 - 8xy + 21xy - 14y^2$ and then combine like terms to come up with the final solution.

Multiplying in rows

A third method for multiplying polynomials looks very similar to multiplying numbers. Consider the problem:

$$\begin{array}{rl}
35 & \\
\underline{\times 27} & \\
245 & \text{Multiply 7 by 5 then 3} \\
\underline{700} & \text{Use 0 for placeholder, multiply 2 by 5 then 3} \\
945 & \text{Add to get Our Solution}
\end{array}$$

World View Note: The first known system that used place values comes from Chinese mathematics, dating back to 190 AD or earlier.

The same process can be done with polynomials. Multiply each term on the bottom with each term on the top.

Example 8.

$$\begin{array}{rl}
(4x+7y)(3x-2y) & \text{Rewrite as vertical problem} \\
4x+7y & \\
\underline{\times\, 3x-2y} & \\
-8xy-14y^2 & \text{Multiply } -2y \text{ by } 7y \text{ then } 4x \\
\underline{12x^2+21xy \qquad\quad} & \text{Multiply } 3x \text{ by } 7y \text{ then } 4x\text{. Line up like terms} \\
12x^2+13xy-14y^2 & \text{Add like terms to get Our Solution}
\end{array}$$

This same process is easily expanded to a problem with more terms.

Example 9.

$$\begin{array}{rl}
(2x-5)(4x^2-7x+3) & \text{Rewrite as vertical problem} \\
4x^3-7x+3 & \text{Put polynomial with most terms on top} \\
\underline{\times\, 2x-5} & \\
-20x^2+35x-15 & \text{Multiply } -5 \text{ by each term} \\
\underline{8x^3-14x^2+6x\qquad} & \text{Multiply } 2x \text{ by each term. Line up like terms} \\
8x^3-34x^2+41x-15 & \text{Add like terms to get our solution}
\end{array}$$

This method of multiplying in rows also works with multiplying a monomial by a polynomial!

Any of the three described methods work to multiply polynomials. It is suggested that you are very comfortable with at least one of these methods as you work through the practice problems. All three methods are shown side by side in the example.

Example 10.

$$(2x-y)(4x-5y)$$

Distribute	**FOIL**	**Rows**
$4x(2x-y)-5y(2x-y)$	$2x(4x)+2x(-5y)-y(4x)-y(-5y)$	$2x-y$
$8x^2-4xy-10xy-5y^2$	$8x^2-10xy-4xy+5y^2$	$\underline{\times\, 4x-5y}$
$8x^2-14xy-5y^2$	$8x^2-14xy+5y^2$	$-10xy+5y^2$
		$\underline{8x^2-4xy\qquad}$
		$8x^2-14xy+5y^2$

When we are multiplying a monomial by a polynomial by a polynomial we can solve by first multiplying the polynomials then distributing the coefficient last. This is shown in the last example.

Example 11.

$$\begin{array}{rl}
3(2x-4)(x+5) & \text{Multiply the binomials, we will use FOIL} \\
3(2x^2+10x-4x-20) & \text{Combine like terms} \\
3(2x^2+6x-20) & \text{Distribute the 3} \\
6x^2+18x-60 & \text{Our Solution}
\end{array}$$

A common error students do is distribute the three at the start into both parenthesis. While we can distribute the 3 into the $(2x-4)$ factor, distributing into both would be wrong. Be careful of this error. This is why it is suggested to multiply the binomials first, then distribute the coeffienct last.

2.5 Practice - Multiply Polynomials

Find each product.

1) $6(p-7)$

2) $4k(8k+4)$

3) $2(6x+3)$

4) $3n^2(6n+7)$

5) $5m^4(4m+4)$

6) $3(4r-7)$

7) $(4n+6)(8n+8)$

8) $(2x+1)(x-4)$

9) $(8b+3)(7b-5)$

10) $(r+8)(4r+8)$

11) $(4x+5)(2x+3)$

12) $(7n-6)(n+7)$

13) $(3v-4)(5v-2)$

14) $(6a+4)(a-8)$

15) $(6x-7)(4x+1)$

16) $(5x-6)(4x-1)$

17) $(5x+y)(6x-4y)$

18) $(2u+3v)(8u-7v)$

19) $(x+3y)(3x+4y)$

20) $(8u+6v)(5u-8v)$

21) $(7x+5y)(8x+3y)$

22) $(5a+8b)(a-3b)$

23) $(r-7)(6r^2-r+5)$

24) $(4x+8)(4x^2+3x+5)$

25) $(6n-4)(2n^2-2n+5)$

26) $(2b-3)(4b^2+4b+4)$

27) $(6x+3y)(6x^2-7xy+4y^2)$

28) $(3m-2n)(7m^2+6mn+4n^2)$

29) $(8n^2+4n+6)(6n^2-5n+6)$

30) $(2a^2+6a+3)(7a^2-6a+1)$

31) $(5k^2+3k+3)(3k^2+3k+6)$

32) $(7u^2+8uv-6v^2)(6u^2+4uv+3v^2)$

33) $3(3x-4)(2x+1)$

34) $5(x-4)(2x-3)$

35) $3(2x+1)(4x-5)$

36) $2(4x+1)(2x-6)$

37) $7(x-5)(x-2)$

38) $5(2x-1)(4x+1)$

39) $6(4x-1)(4x+1)$

40) $3(2x+3)(6x+9)$

Answers to Multiply Polynomials

1) $6p-42$

2) $32k^2+16k$

3) $12x+6$

4) $18n^3+21n^2$

5) $20m^5+20m^4$

6) $12r-21$

7) $32n^2+80n+48$

8) $2x^2-7x-4$

9) $56b^2-19b-15$

10) $4r^2+40r+64$

11) $8x^2+22x+15$

12) $7n^2+43n-42$

13) $15v^2-26v+8$

14) $6a^2-44a-32$

15) $24x^2-22x-7$

16) $20x^2-29x+6$

17) $30x^2-14xy-4y^2$

18) $16u^2+10uv-21v^2$

19) $3x^2+13xy+12y^2$

20) $40u^2-34uv-48v^2$

21) $56x^2+61xy+15y^2$

22) $5a^2-7ab-24b^2$

23) $6r^3-43r^2+12r-35$

24) $16x^3+44x^2+44x+40$

25) $12n^3-20n^2+38n-20$

26) $8b^3-4b^2-4b-12$

27) $36x^3-24x^2y+3xy^2+12y^3$

28) $21m^3+4m^2n-8n^3$

29) $48n^4-16n^3+64n^2-6n+36$

30) $14a^4+30a^3-13a^2-12a+3$

31) $15k^4+24k^3+48k^2+27k+18$

32) $42u^4+76u^3v+17u^2v^2-18v^4$

33) $18x^2-15x-12$

34) $10x^2-55x+60$

35) $24x^2-18x-15$

36) $16x^2-44x-12$

37) $7x^2-49x+70$

38) $40x^2-10x-5$

39) $96x^2-6$

40) $36x^2+108x+81$

2.6 Multiply Special Products

Objective: Recognize and use special product rules of a sum and difference and perfect squares to multiply polynomials.

There are a few shortcuts that we can take when multiplying polynomials. If we can recognize them the shortcuts can help us arrive at the solution much quicker. These shortcuts will also be useful to us as our study of algebra continues.

The first shortcut is often called a **sum and a difference**. A sum and a difference is easily recognized as the numbers and variables are exactly the same, but the sign in the middle is different (one sum, one difference). To illustrate the shortcut consider the following example, multiplied by the distributing method.

Example 1.

$$\begin{array}{rl} (a+b)(a-b) & \text{Distribute } (a+b) \\ a(a+b)-b(a+b) & \text{Distribute } a \text{ and } -b \\ a^2+ab-ab-b^2 & \text{Combine like terms } ab-ab \\ a^2-b^2 & \text{Our Solution} \end{array}$$

The important part of this example is the middle terms subtracted to zero. Rather than going through all this work, when we have a sum and a difference we will jump right to our solution by squaring the first term and squaring the last term, putting a subtraction between them. This is illustrated in the following example

Example 2.

$$\begin{array}{rl} (x-5)(x+5) & \text{Recognize sum and difference} \\ x^2-25 & \text{Square both, put subtraction between. Our Solution} \end{array}$$

This is much quicker than going through the work of multiplying and combining like terms. Often students ask if they can just multiply out using another method and not learn the shortcut. These shortcuts are going to be very useful when we get to factoring polynomials, or reversing the multiplication process. For this reason it is very important to be able to recognize these shortcuts. More examples are shown here.

Example 3.

$$\begin{array}{rl} (3x+7)(3x-7) & \text{Recognize sum and difference} \\ 9x^2-49 & \text{Square both, put subtraction between. Our Solution} \end{array}$$

Example 4.

$$\begin{array}{rl} (2x-6y)(2x+6y) & \text{Recognize sum and difference}\\ 4x^2-36y^2 & \text{Square both, put subtraction between. Our Solution} \end{array}$$

It is interesting to note that while we can multiply and get an answer like a^2-b^2 (with subtraction), it is impossible to multiply real numbers and end up with a product such as a^2+b^2 (with addition).

Another shortcut used to multiply is known as a **perfect square**. These are easy to recognize as we will have a binomial with a 2 in the exponent. The following example illustrates multiplying a perfect square

Example 5.

$$\begin{array}{rl} (a+b)^2 & \text{Squared is same as multiplying by itself}\\ (a+b)(a+b) & \text{Distribute } (a+b)\\ a(a+b)+b(a+b) & \text{Distribute again through final parenthesis}\\ a^2+ab+ab+b^2 & \text{Combine like terms } ab+ab\\ a^2+2\text{ab}+b^2 & \text{Our Solution} \end{array}$$

This problem also helps us find our shortcut for multiplying. The first term in the answer is the square of the first term in the problem. The middle term is 2 times the first term times the second term. The last term is the square of the last term. This can be shortened to square the first, twice the product, square the last. If we can remember this shortcut we can square any binomial. This is illustrated in the following example

Example 6.

$$\begin{array}{rl} (x-5)^2 & \text{Recognize perfect square}\\ x^2 & \text{Square the first}\\ 2(x)(-5)=-10x & \text{Twice the product}\\ (-5)^2=25 & \text{Square the last}\\ x^2-10x+25 & \text{Our Solution} \end{array}$$

Be very careful when we are squaring a binomial to **NOT** distribute the square through the parenthesis. A common error is to do the following: $(x-5)^2=x^2-25$ (or x^2+25). Notice both of these are missing the middle term, $-10x$. This is why it is important to use the shortcut to help us find the correct solution. Another important observation is that the middle term in the solution always has the same sign as the middle term in the problem. This is illustrated in the next examples.

Example 7.

$$\begin{array}{rl}
(2x+5)^2 & \text{Recognize perfect square} \\
(2x)^2 = 4x^2 & \text{Square the first} \\
2(2x)(5) = 20x & \text{Twice the product} \\
5^2 = 25 & \text{Square the last} \\
4x^2 + 20x + 25 & \text{Our Solution}
\end{array}$$

Example 8.

$$\begin{array}{rl}
(3x-7y)^2 & \text{Recognize perfect square} \\
9x^2 - 42\text{xy} + 49y^2 & \text{Square the first, twice the product, square the last. Our Solution}
\end{array}$$

Example 9.

$$\begin{array}{rl}
(5a+9b)^2 & \text{Recognize perfect square} \\
25a^2 + 90\text{ab} + 81b^2 & \text{Square the first, twice the product, square the last. Our Solution}
\end{array}$$

These two formulas will be important to commit to memory. The more familiar we are with them, the easier factoring, or multiplying in reverse, will be. The final example covers both types of problems (two perfect squares, one positive, one negative), be sure to notice the difference between the examples and how each formula is used

Example 10.

$$\begin{array}{ccc}
(4x-7)(4x+7) & (4x+7)^2 & (4x-7)^2 \\
16x^2 - 49 & 16x^2 + 56x + 49 & 16x^2 - 56x + 49
\end{array}$$

World View Note: There are also formulas for higher powers of binomials as well, such as $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. While French mathematician Blaise Pascal often gets credit for working with these expansions of binomials in the 17th century, Chinese mathematicians had been working with them almost 400 years earlier!

2.6 Practice - Multiply Special Products

Find each product.

1) $(x+8)(x-8)$

2) $(a-4)(a+4)$

3) $(1+3p)(1-3p)$

4) $(x-3)(x+3)$

5) $(1-7n)(1+7n)$

6) $(8m+5)(8m-5)$

7) $(5n-8)(5n+8)$

8) $(2r+3)(2r-3)$

9) $(4x+8)(4x-8)$

10) $(b-7)(b+7)$

11) $(4y-x)(4y+x)$

12) $(7a+7b)(7a-7b)$

13) $(4m-8n)(4m+8n)$

14) $(3y-3x)(3y+3x)$

15) $(6x-2y)(6x+2y)$

16) $(1+5n)^2$

17) $(a+5)^2$

18) $(v+4)^2$

19) $(x-8)^2$

20) $(1-6n)^2$

21) $(p+7)^2$

22) $(7k-7)^2$

23) $(7-5n)^2$

24) $(4x-5)^2$

25) $(5m-8)^2$

26) $(3a+3b)^2$

27) $(5x+7y)^2$

28) $(4m-n)^2$

29) $(2x+2y)^2$

30) $(8x+5y)^2$

31) $(5+2r)^2$

32) $(m-7)^2$

33) $(2+5x)^2$

34) $(8n+7)(8n-7)$

35) $(4v-7)\,(4v+7)$

36) $(b+4)(b-4)$

37) $(n-5)(n+5)$

38) $(7x+7)^2$

39) $(4k+2)^2$

40) $(3a-8)(3a+8)$

2.6

Answers to Multiply Special Products

1) x^2-64
2) a^2-16
3) $1-9p^2$
4) x^2-9
5) $1-49n^2$
6) $64m^2-25$
7) $25n^2-64$
8) $4r^2-9$
9) $16x^2-64$
10) b^2-49
11) $16y^2-x^2$
12) $49a^2-49b^2$
13) $16m^2-64n^2$
14) $9y^2-9x^2$
15) $36x^2-4y^2$
16) $1+10n+25n^2$
17) $a^2+10a+25$
18) $v^2+8v+16$
19) $x^2-16x+64$
20) $1-12n+36n^2$
21) $p^2+14p+49$
22) $49k^2-98k+49$
23) $49-70n+25n^2$
24) $16x^2-40x+25$
25) $25m^2-80m+64$
26) $9a^2+18ab+9b^2$
27) $25x^2+70xy+49y^2$
28) $16m^2-8mn+n^2$
29) $4x^2+8xy+4y^2$
30) $64x^2+80xy+25y^2$
31) $25+20r+4r^2$
32) $m^2-14m+49$
33) $4+20x+25x^2$
34) $64n^2-49$
35) $16v^2-49$
36) b^2-16
37) n^2-25
38) $49x^2+98x+49$
39) $16k^2+16k+4$
40) $9a^2-64$

2.7 Divide Polynomials

Objective: Divide polynomials using long division.

Dividing polynomials is a process very similar to long division of whole numbers. But before we look at that, we will first want to be able to master dividing a polynomial by a monomial. The way we do this is very similar to distributing, but the operation we distribute is the division, dividing each term by the monomial and reducing the resulting expression. This is shown in the following examples

Example 1.

$$\frac{9x^5+6x^4-18x^3-24x^2}{3x^2} \quad \text{Divide each term in the numerator by } 3x^2$$

$$\frac{9x^5}{3x^2}+\frac{6x^4}{3x^2}-\frac{18x^3}{3x^2}-\frac{24x^2}{3x^2} \quad \text{Reduce each fraction, subtracting exponents}$$

$$3x^3+2x^2-6x-8 \quad \text{Our Solution}$$

Example 2.

$$\frac{8x^3+4x^2-2x+6}{4x^2} \quad \text{Divide each term in the numerator by } 4x^2$$

$$\frac{8x^3}{4x^2}+\frac{4x^2}{4x^2}-\frac{2x}{4x^2}+\frac{6}{4x^2} \quad \text{Reduce each fraction, subtracting exponents}$$

Remember negative exponents are moved to denominator

$$2x+1-\frac{1}{2x}+\frac{3}{2x^2} \quad \text{Our Solution}$$

The previous example illustrates that sometimes we will have fractions in our solution, as long as they are reduced this will be correct for our solution. Also interesting in this problem is the second term $\frac{4x^2}{4x^2}$ divided out completely. Remember that this means the reduced answer is 1 not 0.

Long division is required when we divide by more than just a monomial. Long division with polynomials works very similar to long division with whole numbers.

An example is given to review the (general) steps that are used with whole numbers that we will also use with polynomials

Example 3.

$$\begin{array}{rl}
4|\overline{631} & \text{Divide front numbers: } \dfrac{6}{4} = 1\ldots \\
\mathbf{1} & \\
4|\overline{631} & \text{Multiply this number by divisor: } 1 \cdot 4 = 4 \\
\underline{\mathbf{-\,4}} & \text{Change the sign of this number (make it subtract) and combine} \\
\mathbf{23} & \text{Bring down next number}
\end{array}$$

$$\begin{array}{rl}
1\mathbf{5} & \text{Repeat, divide front numbers: } \dfrac{23}{4} = 5\ldots \\
4|\overline{631} & \\
\underline{-\,4} & \\
23 & \text{Multiply this number by divisor: } 5 \cdot 4 = 20 \\
\underline{\mathbf{-\,20}} & \text{Change the sign of this number (make it subtract) and combine} \\
\mathbf{31} & \text{Bring down next number}
\end{array}$$

$$\begin{array}{rl}
15\mathbf{7} & \text{Repeat, divide front numbers: } \dfrac{31}{4} = 7\ldots \\
4|\overline{631} & \\
\underline{-\,4} & \\
23 & \\
\underline{-\,20} & \\
31 & \text{Multiply this number by divisor: } 7 \cdot 4 = 28 \\
\underline{\mathbf{-\,28}} & \text{Change the sign of this number (make it subtract) and combine} \\
\mathbf{3} & \text{We will write our remainder as } a \text{ fraction, over the divisor, added to the end}
\end{array}$$

$$\begin{array}{rl}
157\dfrac{3}{4} & \text{Our Solution}
\end{array}$$

This same process will be used to multiply polynomials. The only difference is we will replace the word "number" with the word "term"

Dividing Polynomials

1. Divide front terms

2. Multiply this term by the divisor

3. Change the sign of the terms and combine

4. Bring down the next term

5. Repeat

Step number 3 tends to be the one that students skip, not changing the signs of the terms would be equivalent to adding instead of subtracting on long division with whole numbers. Be sure not to miss this step! This process is illustrated in the following two examples.

Example 4.

$\dfrac{3x^3 - 5x^2 - 32x + 7}{x - 4}$ Rewrite problem as long division

$x - 4\overline{)3x^3 - 5x^2 - 32x + 7}$ Divide front terms: $\dfrac{3x^3}{x} = 3x^2$

$$\begin{array}{rl}
\mathbf{3x^2} & \\
x - 4\overline{)3x^3 - 5x^2 - 32x + 7} & \text{Multiply this term by divisor: } 3x^2(x-4) = 3x^3 - 12x^2 \\
\underline{\mathbf{- 3x^3 + 12x^2} } & \text{Change the signs and combine} \\
\mathbf{7x^2 - 32x} & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rl}
3x^2 + \mathbf{7x} & \text{Repeat, divide front terms: } \dfrac{7x^2}{x} = 7x \\
x - 4\overline{)3x^3 - 5x^2 - 32x + 7} & \\
\underline{- 3x^3 + 12x^2 } & \\
7x^2 - 32x & \text{Multiply this term by divisor: } 7x(x-4) = 7x^2 - 28x \\
\underline{\mathbf{- 7x^2 + 28x} } & \text{Change the signs and combine} \\
\mathbf{- 4x + 7} & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rl}
3x^2 + 7x \mathbf{- 4} & \text{Repeat, divide front terms: } \dfrac{-4x}{x} = -4 \\
x - 4\overline{)3x^3 - 5x^2 - 32x + 7} & \\
\underline{- 3x^3 + 12x^2 } & \\
7x^2 - 32x & \\
\underline{- 7x^2 + 28x } & \\
- 4x + 7 & \text{Multiply this term by divisor: } -4(x-4) = -4x + 16 \\
\underline{\mathbf{+ 4x - 16}} & \text{Change the signs and combine} \\
\mathbf{- 9} & \text{Remainder put over divisor and subtracted (due to negative)}
\end{array}$$

$3x^2+7x-4-\frac{9}{x-4}$ Our Solution

Example 5.

$\frac{6x^3-8x^2+10x+103}{2x+4}$ Rewrite problem as long division

$2x+4|\overline{6x^3-8x^2+10x+103}$ Divide front terms: $\frac{6x^3}{2x}=3x^2$

$$
\begin{array}{rl}
\mathbf{3x^2} & \\
2x+4|\overline{6x^3-8x^2+10x+103} & \text{Multiply term by divisor: } 3x^2(2x+4)=6x^3+12x^2 \\
\underline{\mathbf{-6x^3-12x^2}} & \text{Change the signs and combine} \\
\mathbf{-20x^2+10x} & \text{Bring down the next term}
\end{array}
$$

$$
\begin{array}{rl}
3x^2\mathbf{-10x} & \\
2x+4|\overline{6x^3-8x^2+10x+103} & \text{Repeat, divide front terms: } \frac{-20x^2}{2x}=-10x \\
\underline{-6x^3-12x^2} & \text{Multiply this term by divisor:} \\
-20x^2+10x & -10x(2x+4)=-20x^2-40x \\
\underline{\mathbf{+20x^2+40x}} & \text{Change the signs and combine} \\
\mathbf{50x+103} & \text{Bring down the next term}
\end{array}
$$

$$
\begin{array}{rl}
3x^2-10x\mathbf{+25} & \\
2x+4|\overline{6x^3-8x^2+10x+103} & \text{Repeat, divide front terms: } \frac{50x}{2x}=25 \\
\underline{-6x^3-12x^2} & \\
-20x^2+10x & \\
\underline{+20x^2+40x} & \\
50x+103 & \text{Multiply this term by divisor: } 25(2x+4)=50x+100 \\
\underline{\mathbf{-50x-100}} & \text{Change the signs and combine} \\
3 & \text{Remainder is put over divsor and added (due to positive)}
\end{array}
$$

$3x^2-10x+25+\frac{3}{2x+4}$ Our Solution

In both of the previous example the dividends had the exponents on our variable counting down, no exponent skipped, third power, second power, first power, zero power (remember $x^0=1$ so there is no variable on zero power). This is very important in long division, the variables must count down and no exponent can be skipped. If they don't count down we must put them in order. If an exponent is skipped we will have to add a term to the problem, with zero for its coefficient. This is demonstrated in the following example.

Example 6.

$$\frac{2x^3+42-4x}{x+3} \quad \text{Reorder dividend, need } x^2 \text{ term, add } 0x^2 \text{ for this}$$

$$x+3\overline{)2x^3+\mathbf{0x^2}-4x+42} \quad \text{Divide front terms: } \frac{2x^3}{x}=2x^2$$

$$\begin{array}{rl} \mathbf{2x^2} & \\ x+3\overline{)2x^3+0x^2-4x+42} & \text{Multiply this term by divisor: } 2x^2(x+3)=2x^3+6x^2 \\ \underline{\mathbf{-2x^3-6x^2}} & \text{Change the signs and combine} \\ \mathbf{-6x^2-4x} & \text{Bring down the next term} \end{array}$$

$$\begin{array}{rl} 2x^2\mathbf{-6x} & \\ x+3\overline{)2x^3+0x^2-4x+42} & \text{Repeat, divide front terms: } \frac{-6x^2}{x}=-6x \\ \underline{-2x^3-6x^2} & \\ -6x^2-4x & \text{Multiply this term by divisor: } -6x(x+3)=-6x^2-18x \\ \underline{\mathbf{+6x^2+18x}} & \text{Change the signs and combine} \\ \mathbf{14x+42} & \text{Bring down the next term} \end{array}$$

$$\begin{array}{rl} 2x^2-6x\mathbf{+14} & \\ x+3\overline{)2x^3+0x^2-4x+42} & \text{Repeat, divide front terms: } \frac{14x}{x}=14 \\ \underline{-2x^3-6x^2} & \\ -6x^2-4x & \\ \underline{+6x^2+18x} & \\ 14x+42 & \text{Multiply this term by divisor: } 14(x+3)=14x+42 \\ \underline{\mathbf{-14x-42}} & \text{Change the signs and combine} \\ 0 & \text{No remainder} \end{array}$$

$$2x^2-6x+14 \quad \text{Our Solution}$$

It is important to take a moment to check each problem to verify that the exponents count down and no exponent is skipped. If so we will have to adjust the problem. Also, this final example illustrates, just as in regular long division, sometimes we have no remainder in a problem.

World View Note: Paolo Ruffini was an Italian Mathematician of the early 19th century. In 1809 he was the first to describe a process called synthetic division which could also be used to divide polynomials.

2.7 Practice - Divide Polynomials

Divide.

1) $\frac{20x^4+x^3+2x^2}{4x^3}$

3) $\frac{20n^4+n^3+40n^2}{10n}$

5) $\frac{12x^4+24x^3+3x^2}{6x}$

7) $\frac{10n^4+50n^3+2n^2}{10n^2}$

9) $\frac{x^2-2x-71}{x+8}$

11) $\frac{n^2+13n+32}{n+5}$

13) $\frac{v^2-2v-89}{v-10}$

15) $\frac{a^2-4a-38}{a-8}$

17) $\frac{45p^2+56p+19}{9p+4}$

19) $\frac{10x^2-32x+9}{10x-2}$

21) $\frac{4r^2-r-1}{4r+3}$

23) $\frac{n^2-4}{n-2}$

25) $\frac{27b^2+87b+35}{3b+8}$

27) $\frac{4x^2-33x+28}{4x-5}$

29) $\frac{a^3+15a^2+49a-55}{a+7}$

31) $\frac{x^3-26x-41}{x+4}$

33) $\frac{3n^3+9n^2-64n-68}{n+6}$

35) $\frac{x^3-46x+22}{x+7}$

37) $\frac{9p^3+45p^2+27p-5}{9p+9}$

39) $\frac{r^3-r^2-16r+8}{r-4}$

41) $\frac{12n^3+12n^2-15n-4}{2n+3}$

43) $\frac{4v^3-21v^2+6v+19}{4v+3}$

2) $\frac{5x^4+45x^3+4x^2}{9x}$

4) $\frac{3k^3+4k^2+2k}{8k}$

6) $\frac{5p^4+16p^3+16p^2}{4p}$

8) $\frac{3m^4+18m^3+27m^2}{9m^2}$

10) $\frac{r^2-3r-53}{r-9}$

12) $\frac{b^2-10b+16}{b-7}$

14) $\frac{x^2+4x-26}{x+7}$

16) $\frac{x^2-10x+22}{x-4}$

18) $\frac{48k^2-70k+16}{6k-2}$

20) $\frac{n^2+7n+15}{n+4}$

22) $\frac{3m^2+9m-9}{3m-3}$

24) $\frac{2x^2-5x-8}{2x+3}$

26) $\frac{3v^2-32}{3v-9}$

28) $\frac{4n^2-23n-38}{4n+5}$

30) $\frac{8k^3-66k^2+12k+37}{k-8}$

32) $\frac{x^3-16x^2+71x-56}{x-8}$

34) $\frac{k^3-4k^2-6k+4}{k-1}$

36) $\frac{2n^3+21n^2+25n}{2n+3}$

38) $\frac{8m^3-57m^2+42}{8m+7}$

40) $\frac{2x^3+12x^2+4x-37}{2x+6}$

42) $\frac{24b^3-38b^2+29b-60}{4b-7}$

Answers to Divide Polynomials

1) $5x+\frac{1}{4}+\frac{1}{2x}$

2) $\frac{5x^3}{9}+5x^2+\frac{4x}{9}$

3) $2n^3+\frac{n^2}{10}+4n$

4) $\frac{3k^2}{8}+\frac{k}{2}+\frac{1}{4}$

5) $2x^3+4x^2+\frac{x}{2}$

6) $\frac{5p^3}{4}+4p^2+4p$

7) $n^2+5n+\frac{1}{5}$

8) $\frac{m^2}{3}+2m+3$

9) $x-10+\frac{9}{x+8}$

10) $r+6+\frac{1}{r-9}$

11) $n+8-\frac{8}{n+5}$

12) $b-3-\frac{5}{b-7}$

13) $v+8-\frac{9}{v-10}$

14) $x-3-\frac{5}{x+7}$

15) $a+4-\frac{6}{a-8}$

16) $x-6-\frac{2}{x-4}$

17) $5p+4+\frac{3}{9p+4}$

18) $8k-9-\frac{1}{3k-1}$

19) $x-3+\frac{3}{10x-2}$

20) $n+3+\frac{3}{n+4}$

21) $r-1+\frac{2}{4x+3}$

22) $m+4+\frac{1}{m-1}$

23) $n+2$

24) $x-4+\frac{4}{2x+3}$

25) $9b+5-\frac{5}{3b+8}$

26) $v+3-\frac{5}{3v-9}$

27) $x-7-\frac{7}{4x-5}$

28) $n-7-\frac{3}{4n+5}$

29) $a^2+8a-7-\frac{6}{a+7}$

30) $8k^2-2k-4+\frac{5}{k-8}$

31) $x^2-4x-10-\frac{1}{x+4}$

32) x^2-8x+7

33) $3n^2-9n-10-\frac{8}{n+6}$

34) $k^2-3k-9-\frac{5}{k-1}$

35) $x^2-7x+3+\frac{1}{x+7}$

36) $n^2+9n-1+\frac{3}{2n+3}$

37) $p^2+4p-1+\frac{4}{9p+9}$

38) $m^2-8m+7-\frac{7}{8m+7}$

39) $r^2+3r-4-\frac{8}{r-4}$

40) $x^2+3x-7+\frac{5}{2x+6}$

41) $6n^2-3n-3+\frac{5}{2n+3}$

42) $6b^2+b+9+\frac{3}{4b-7}$

43) $v^2-6v+6+\frac{1}{4v+3}$

Chapter 3 - Factoring
3.1 Greatest Common Factor

Objective: Find the greatest common factor of a polynomial and factor it out of the expression.

The opposite of multiplying polynomials together is factoring polynomials. There are many benifits of a polynomial being factored. We use factored polynomials to help us solve equations, learn behaviors of graphs, work with fractions and more. Because so many concepts in algebra depend on us being able to factor polynomials it is very important to have very strong factoring skills.

In this lesson we will focus on factoring using the greatest common factor or GCF of a polynomial. When we multiplied polynomials, we multiplied monomials by polynomials by distributing, solving problems such as $4x^2(2x^2 - 3x + 8) = 8x^4 - 12x^3 + 32x$. In this lesson we will work the same problem backwards. We will start with $8x^2 - 12x^3 + 32x$ and try and work backwards to the $4x^2(2x - 3x + 8)$.

To do this we have to be able to first identify what is the GCF of a polynomial. We will first introduce this by looking at finding the GCF of several numbers. To find a GCF of sevearal numbers we are looking for the largest number that can be divided by each of the numbers. This can often be done with quick mental math and it is shown in the following example

Example 1.

$$\begin{array}{rl} \text{Find the GCF of } 15, 24, \text{and } 27 & \\ \frac{15}{3} = 5,\ \frac{24}{3} = 6,\ \frac{27}{3} = 9 & \text{Each of the numbers can be divided by 3} \\ \text{GCF} = 3 & \text{Our Solution} \end{array}$$

When there are variables in our problem we can first find the GCF of the numbers using mental math, then we take any variables that are in common with each term, using the lowest exponent. This is shown in the next example

Example 2.

$$\begin{aligned}
&\text{GCF of } 24x^4y^2z, 18x^2y^4, \text{and } 12x^3yz^5 \\
\frac{24}{6}=4,\ \frac{18}{6}=3,\ \frac{12}{6}&=2 && \text{Each number can be divided by 6} \\
x^2y & && x \text{ and } y \text{ are in all 3, using lowest exponets} \\
\text{GCF} = 6x^2y & && \text{Our Solution}
\end{aligned}$$

To factor out a GCF from a polynomial we first need to identify the GCF of all the terms, this is the part that goes in front of the parenthesis, then we divide each term by the GCF, the answer is what is left inside the parenthesis. This is shown in the following examples

Example 3.

$$\begin{aligned}
4x^2-20x+16 & && \text{GCF is 4, divide each term by 4} \\
\frac{4x^2}{4}=x^2,\ \frac{-20x}{4}=-5x,\ \frac{16}{4}=4 & && \text{This is what is left inside the parenthesis} \\
4(x^2-5x+4) & && \text{Our Solution}
\end{aligned}$$

With factoring we can always check our solutions by multiplying (distributing in this case) out the answer and the solution should be the original equation.

Example 4.

$$\begin{aligned}
25x^4-15x^3+20x^2 & && \text{GCF is } 5x^2\text{, divide each term by this} \\
\frac{25x^4}{5x^2}=5x^2,\ \frac{-15x^3}{5x^2}=-3x,\ \frac{20x^2}{5x^2}=4 & && \text{This is what is left inside the parenthesis} \\
5x^2(5x^2-3x+4) & && \text{Our Solution}
\end{aligned}$$

Example 5.

$$\begin{aligned}
3x^3y^2z+5x^4y^3z^5-4xy^4 & && \text{GCF is } xy^2\text{, divide each term by this} \\
\frac{3x^3y^2z}{xy^2}=3x^2z, \frac{5x^4y^3z^5}{xy^2}=5x^3yz^5, \frac{-4xy^4}{xy^2}=-4y^2 & && \text{This is what is left in parenthesis} \\
xy^2(3x^2z+5x^3yz^5-4y^2) & && \text{Our Solution}
\end{aligned}$$

World View Note: The first recorded algorithm for finding the greatest common factor comes from Greek mathematician Euclid around the year 300 BC!

Example 6.

$$\begin{array}{rl} 21x^3+14x^2+7x & \text{GCF is } 7x\text{, divide each term by this} \\ \dfrac{21x^3}{7x}=3x^2,\ \dfrac{14x^2}{7x}=2x,\ \dfrac{7x}{7x}=1 & \text{This is what is left inside the parenthesis} \\ 7x(3x^2+2x+1) & \text{Our Solution} \end{array}$$

It is important to note in the previous example, that when the GCF was $7x$ and $7x$ was one of the terms, dividing gave an answer of 1. Students often try to factor out the $7x$ and get zero which is incorrect, factoring will never make terms dissapear. Anything divided by itself is 1, be sure to not forget to put the 1 into the solution.

Often the second line is not shown in the work of factoring the GCF. We can simply identify the GCF and put it in front of the parenthesis as shown in the following two examples.

Example 7.

$$\begin{array}{rl} 12x^5y^2-6x^4y^4+8x^3y^5 & \text{GCF is } 2x^3y^2\text{, put this in front of parenthesis and divide} \\ 2x^3y^2(6x^2-3xy^2+4y^3) & \text{Our Solution} \end{array}$$

Example 8.

$$\begin{array}{rl} 18a^4b^3-27a^3b^3+9a^2b^3 & \text{GCF is } 9a^2b^3\text{, divide each term by this} \\ 9a^2b^3(2a^2-3a+1) & \text{Our Solution} \end{array}$$

Again, in the previous problem, when dividing $9a^2b^3$ by itself, the answer is 1, not zero. Be very careful that each term is accounted for in your final solution.

3.1 Practice - Greatest Common Factor

Factor the common factor out of each expression.

1) $9+8b^2$

2) $x-5$

3) $45x^2-25$

4) $1+2n^2$

5) $56-35p$

6) $50x-80y$

7) $7\text{ab}-35a^2b$

8) $27x^2y^5-72x^3y^2$

9) $-3a^2b+6a^3b^2$

10) $8x^3y^2+4x^3$

11) $-5x^2-5x^3-15x^4$

12) $-32n^9+32n^6+40n^5$

13) $20x^4-30x+30$

14) $21p^6+30p^2+27$

15) $28m^4+40m^3+8$

16) $-10x^4+20x^2+12x$

17) $30b^9+5ab-15a^2$

18) $27y^7+12y^2x+9y^2$

19) $-48a^2b^2-56a^3b-56a^5b$

20) $30m^6+15mn^2-25$

21) $20x^8y^2z^2+15x^5y^2z+35x^3y^3z$

22) $3p+12q-15q^2r^2$

23) $50x^2y+10y^2+70xz^2$

24) $30y^4z^3x^5+50y^4z^5-10y^4z^3x$

25) $30qpr-5qp+5q$

26) $28b+14b^2+35b^3+7b^5$

27) $-18n^5+3n^3-21n+3$

28) $30a^8+6a^5+27a^3+21a^2$

29) $-40x^{11}-20x^{12}+50x^{13}-50x^{14}$

30) $-24x^6-4x^4+12x^3+4x^2$

31) $-32mn^8+4m^6n+12mn^4+16mn$

32) $-10y^7+6y^{10}-4y^{10}x-8y^8x$

3.1

Answers - Greatest Common Factor

1) $9+8b^2$

2) $x-5$

3) $5(9x^2-5)$

4) $1+2n^2$

5) $7(8-5p)$

6) $10(5x-8y)$

7) $7ab(1-5a)$

8) $9x^2y^2(3y^3-8x)$

9) $3a^2b(-1+2ab)$

10) $4x^3(2y^2+1)$

11) $-5x^2(1+x+3x^2)$

12) $8n^5(-4n^4+4n+5)$

13) $10(2x^4-3x+3)$

14) $3(7p^6+10p^2+9)$

15) $4(7m^4+10m^3+2)$

16) $2x(-5x^3+10x+6)$

17) $5(6b^9+ab-3a^2)$

18) $3y^2(9y^5+4x+3)$

19) $-8a^2b(6b+7a+7a^3)$

20) $5(6m^6+3mn^2-5)$

21) $5x^3y^2z(4x^5z+3x^2+7y)$

22) $3(p+4q-5q^2r^2)$

23) $10(5x^2y+y^2+7xz^2)$

24) $10y^4z^3(3x^5+5z^2-x)$

25) $5q(6pr-p+1)$

26) $7b(4+2b+5b^2+b^4)$

27) $3(-6n^5+n^3-7n+1)$

28) $3a^2(10a^6+2a^3+9a+7)$

29) $10x^{11}(-4-2x+5x^2-5x^3)$

30) $4x^2(-6x^4-x^2+3x+1)$

31) $4mn(-8n^7+m^5+3n^3+4)$

32) $2y^7(-5+3y^3-2xy^3-4xy)$

3.2 Grouping

Objective: Factor polynomials with four terms using grouping.

The first thing we will always do when factoring is try to factor out a GCF. This GCF is often a monomial like in the problem $5xy + 10xz$ the GCF is the monomial $5x$, so we would have $5x(y + 2z)$. However, a GCF does not have to be a monomial, it could be a binomial. To see this, consider the following two example.

Example 1.

$$\begin{array}{rl} 3ax - 7bx & \text{Both have } x \text{ in common, factor it out} \\ x(3a - 7b) & \text{Our Solution} \end{array}$$

Now the same problem, but instead of x we have $(2a + 5b)$.

Example 2.

$$\begin{array}{rl} 3a(2a + 5b) - 7b(2a + 5b) & \text{Both have } (2a + 5b) \text{ in common, factor it out} \\ (2a + 5b)(3a - 7b) & \text{Our Solution} \end{array}$$

In the same way we factored out a GCF of x we can factor out a GCF which is a binomial, $(2a + 5b)$. This process can be extended to factor problems where there is no GCF to factor out, or after the GCF is factored out, there is more factoring that can be done. Here we will have to use another strategy to factor. We will use a process known as grouping. Grouping is how we will factor if there are four terms in the problem. Remember, factoring is like multiplying in reverse, so first we will look at a multiplication problem and then try to reverse the process.

Example 3.

$$\begin{array}{rl} (2a + 3)(5b + 2) & \text{Distribute } (2a + 3) \text{ into second parenthesis} \\ 5b(2a + 3) + 2(2a + 3) & \text{Distribute each monomial} \\ 10\text{ab} + 15b + 4a + 6 & \text{Our Solution} \end{array}$$

The solution has four terms in it. We arrived at the solution by looking at the two parts, $5b(2a + 3)$ and $2(2a + 3)$. When we are factoring by grouping we will always divide the problem into two parts, the first two terms and the last two terms. Then we can factor the GCF out of both the left and right sides. When we do this our hope is what is left in the parenthesis will match on both the left and right. If they match we can pull this matching GCF out front, putting the rest in parenthesis and we will be factored. The next example is the same problem worked backwards, factoring instead of multiplying.

Example 4.

$$\begin{array}{rl} 10\text{ab} + 15b + 4a + 6 & \text{Split problem into two groups} \\ \boxed{10\text{ab} + 15b \,\big|\, + 4a + 6} & \text{GCF on left is } 5b\text{, on the right is } 2 \\ \boxed{5b(2a+3) \,\big|\, + 2(2a+3)} & (2a+3) \text{ is the same! Factor out this GCF} \\ (2a+3)(5b+2) & \text{Our Solution} \end{array}$$

The key for grouping to work is after the GCF is factored out of the left and right, the two binomials must match exactly. If there is any difference between the two we either have to do some adjusting or it can't be factored using the grouping method. Consider the following example.

Example 5.

$$\begin{array}{rl} 6x^2 + 9xy - 14x - 21y & \text{Split problem into two groups} \\ \boxed{6x^2 + 9xy \,\big|\, - 14x - 21y} & \text{GCF on left is } 3x\text{, on right is } 7 \\ \boxed{3x(2x+3y) \,\big|\, + 7(-2x-3y)} & \text{The signs in the parenthesis don't match!} \end{array}$$

when the signs don't match on both terms we can easily make them match by factoring the opposite of the GCF on the right side. Instead of 7 we will use -7. This will change the signs inside the second parenthesis.

$$\begin{array}{rl} \boxed{3x(2x+3y) \,\big|\, - 7(2x+3y)} & (2x+3y) \text{ is the same! Factor out this GCF} \\ (2x+3y)(3x-7) & \text{Our Solution} \end{array}$$

Often we can recognize early that we need to use the opposite of the GCF when factoring. If the first term of the first binomial is positive in the problem, we will also want the first term of the second binomial to be positive. If it is negative then we will use the opposite of the GCF to be sure they match.

Example 6.

$$\begin{array}{rl} 5xy - 8x - 10y + 16 & \text{Split the problem into two groups} \\ \boxed{5xy - 8x \,\big|\, - 10y + 16} & \text{GCF on left is } x\text{, on right we need a negative,} \\ & \text{so we use } -2 \\ \boxed{x(5y-8) \,\big|\, - 2(5y-8)} & (5y-8) \text{ is the same! Factor out this GCF} \\ (5y-8)(x-2) & \text{Our Solution} \end{array}$$

Sometimes when factoring the GCF out of the left or right side there is no GCF to factor out. In this case we will use either the GCF of 1 or -1. Often this is all we need to be sure the two binomials match.

Example 7.

$12ab-14a-6b+7$	Split the problem into two groups
$\boxed{12\text{ab}-14a}\boxed{-6b+7}$	GCF on left is $2a$, on right, no GCF, use -1
$\boxed{2a(6b-7)}\boxed{-1(6b-7)}$	$(6b-7)$ is the same! Factor out this GCF
$(6b-7)(2a-1)$	Our Solution

Example 8.

$6x^3-15x^2+2x-5$	Split problem into two groups
$\boxed{6x^3-15x^2}\boxed{+2x-5}$	GCF on left is $3x^2$, on right, no GCF, use 1
$\boxed{3x^2(2x-5)}\boxed{+1(2x-5)}$	$(2x-5)$ is the same! Factor out this GCF
$(2x-5)(3x^2+1)$	Our Solution

Another problem that may come up with grouping is after factoring out the GCF on the left and right, the binomials don't match, more than just the signs are different. In this case we may have to adjust the problem slightly. One way to do this is to change the order of the terms and try again. To do this we will move the second term to the end of the problem and see if that helps us use grouping.

Example 9.

$4a^2-21b^3+6ab-14ab^2$	Split the problem into two groups
$\boxed{4a^2-21b^3}\boxed{+6ab-14ab^2}$	GCF on left is 1, on right is $2ab$
$\boxed{1(4a^2-21b^3)}\boxed{+2ab(3-7b)}$	Binomials don't match! Move second term to end
$4a^2+6ab-14ab^2-21b^3$	Start over, split the problem into two groups
$\boxed{4a^2+6\text{ab}}\boxed{-14ab^2-21b^3}$	GCF on left is $2a$, on right is $-7b^2$
$\boxed{2a(2a+3b)}\boxed{-7b^2(2a+3b)}$	$(2a+3b)$ is the same! Factor out this GCF
$(2a+3b)(2a-7b^2)$	Our Solution

When rearranging terms the problem can still be out of order. Sometimes after factoring out the GCF the terms are backwards. There are two ways that this can happen, one with addition, one with subtraction. If it happens with addition, for example the binomials are $(a+b)$ and $(b+a)$, we don't have to do any extra work. This is because addition is the same in either order $(5+3=3+5=8)$.

Example 10.

$$\begin{array}{rl} 7+y-3xy-21x & \text{Split the problem into two groups} \\ \boxed{7+y}\boxed{-3xy-21x} & \text{GCF on left is } 1, \text{ on the right is } -3x \\ \boxed{1(7+y)}\boxed{-3x(y+7)} & y+7 \text{ and } 7+y \text{ are the same, use either one} \\ (y+7)(1-3x) & \text{Our Solution} \end{array}$$

However, if the binomial has subtraction, then we need to be a bit more careful. For example, if the binomials are $(a-b)$ and $(b-a)$, we will factor out the opposite of the GCF on one part, usually the second. Notice what happens when we factor out -1.

Example 11.

$$\begin{array}{rl} (b-a) & \text{Factor out } -1 \\ -1(-b+a) & \text{Addition can be in either order, switch order} \\ -1(a-b) & \text{The order of the subtraction has been switched!} \end{array}$$

Generally we won't show all the above steps, we will simply factor out the opposite of the GCF and switch the order of the subtraction to make it match the other binomial.

Example 12.

$$\begin{array}{rl} 8xy-12y+15-10x & \text{Split the problem into two groups} \\ \boxed{8xy-12y}\boxed{15-10x} & \text{GCF on left is } 4y, \text{ on right, } 5 \\ \boxed{4y(2x-3)}\boxed{+5(3-2x)} & \text{Need to switch subtraction order, use } -5 \text{ in middle} \\ \boxed{4y(2y-3)}\boxed{-5(2x-3)} & \text{Now } 2x-3 \text{ match on both! Factor out this GCF} \\ (2x-3)(4y-5) & \text{Our Solution} \end{array}$$

World View Note: Sofia Kovalevskaya of Russia was the first woman on the editorial staff of a mathematical journal in the late 19th century. She also did research on how the rings of Saturn rotated.

3.2 Practice - Grouping

Factor each completely.

1) $40r^3 - 8r^2 - 25r + 5$
2) $35x^3 - 10x^2 - 56x + 16$
3) $3n^3 - 2n^2 - 9n + 6$
4) $14v^3 + 10v^2 - 7v - 5$
5) $15b^3 + 21b^2 - 35b - 49$
6) $6x^3 - 48x^2 + 5x - 40$
7) $3x^3 + 15x^2 + 2x + 10$
8) $28p^3 + 21p^2 + 20p + 15$
9) $35x^3 - 28x^2 - 20x + 16$
10) $7n^3 + 21n^2 - 5n - 15$
11) $7xy - 49x + 5y - 35$
12) $42r^3 - 49r^2 + 18r - 21$
13) $32xy + 40x^2 + 12y + 15x$
14) $15ab - 6a + 5b^3 - 2b^2$
15) $16xy - 56x + 2y - 7$
16) $3mn - 8m + 15n - 40$
17) $2xy - 8x^2 + 7y^3 - 28y^2x$
18) $5mn + 2m - 25n - 10$
19) $40xy + 35x - 8y^2 - 7y$
20) $8xy + 56x - y - 7$
21) $32uv - 20u + 24v - 15$
22) $4uv + 14u^2 + 12v + 42u$
23) $10xy + 30 + 25x + 12y$
24) $24xy + 25y^2 - 20x - 30y^3$
25) $3uv + 14u - 6u^2 - 7v$
26) $56ab + 14 - 49a - 16b$
27) $16xy - 3x - 6x^2 + 8y$

Answers - Grouping

1) $(8r^2-5)(5r-1)$
2) $(5x^2-8)(7x-2)$
3) $(n^2-3)(3n-2)$
4) $(2v^2-1)(7v+5)$
5) $(3b^2-7)(5b+7)$
6) $(6x^2+5)(x-8)$
7) $(3x^2+2)(x+5)$
8) $(7p^2+5)(4p+3)$
9) $(7x^2-4)(5x-4)$
10) $(7n^2-5)(n+3)$
11) $(7x+5)(y-7)$
12) $(7r^2+3)(6r-7)$
13) $(8x+3)(4y+5x)$
14) $(3a+b^2)(5b-2)$
15) $(8x+1)(2y-7)$
16) $(m+5)(3n-8)$
17) $(2x+7y^2)(y-4x)$
18) $(m-5)(5n+2)$
19) $(5x-y)(8y+7)$
20) $(8x-1)(y+7)$
21) $(4u+3)(8v-5)$
22) $2(u+3)(2v+7u)$
23) $(5x+6)(2y+5)$
24) $(4x-5y^2)(6y-5)$
25) $(3u-7)(v-2u)$
26) $(7a-2)(8b-7)$
27) $(2x+1)(8y-3x)$

3.3 Trinomials where a = 1

Objective: Factor trinomials where the coefficient of x^2 is one.

Factoring with three terms, or trinomials, is the most important type of factoring to be able to master. As factoring is multiplication backwards we will start with a multipication problem and look at how we can reverse the process.

Example 1.

$$\begin{array}{rl} (x+6)(x-4) & \text{Distribute } (x+6) \text{ through second parenthesis} \\ x(x+6)-4(x+6) & \text{Distribute each monomial through parenthesis} \\ x^2+6x-4x-24 & \text{Combine like terms} \\ x^2+2x-24 & \text{Our Solution} \end{array}$$

You may notice that if you reverse the last three steps the process looks like grouping. This is because it is grouping! The GCF of the left two terms is x and the GCF of the second two terms is -4. The way we will factor trinomials is to make them into a polynomial with four terms and then factor by grouping. This is shown in the following example, the same problem worked backwards

Example 2.

$$\begin{array}{rl} x^2+2x-24 & \text{Split middle term into } +6x-4x \\ x^2+6x-4x-24 & \text{Grouping: GCF on left is } x\text{, on right is } -4 \\ x(x+6)-4(x+6) & (x+6) \text{ is the same, factor out this GCF} \\ (x+6)(x-4) & \text{Our Solution} \end{array}$$

The trick to make these problems work is how we split the middle term. Why did we pick $+6x-4x$ and not $+5x-3x$? The reason is because $6x-4x$ is the only combination that works! So how do we know what is the one combination that works? To find the correct way to split the middle term we will use what is called the ac method. In the next lesson we will discuss why it is called the ac method. The way the ac method works is we find a pair of numers that multiply to a certain number and add to another number. Here we will try to multiply to get the last term and add to get the coefficient of the middle term. In the previous example that would mean we wanted to multiply to -24 and add to $+2$. The only numbers that can do this are 6 and -4 ($6\cdot-4=-24$ and $6+(-4)=2$). This process is shown in the next few examples

Example 3.

$$\begin{array}{rl} x^2+9x+18 & \text{Want to multiply to 18, add to 9} \\ x^2+6x+3x+18 & \text{6 and 3, split the middle term} \\ x(x+6)+3(x+6) & \text{Factor by grouping} \\ (x+6)(x+3) & \text{Our Solution} \end{array}$$

Example 4.

$$\begin{array}{rl} x^2-4x+3 & \text{Want to multiply to 3, add to} -4 \\ x^2-3x-x+3 & -3 \text{ and } -1\text{, split the middle term} \\ x(x-3)-1(x-3) & \text{Factor by grouping} \\ (x-3)(x-1) & \text{Our Solution} \end{array}$$

Example 5.

$$\begin{array}{rl} x^2-8x-20 & \text{Want to multiply to} -20\text{, add to} -8 \\ x^2-10x+2x-20 & -10 \text{ and 2, split the middle term} \\ x(x-10)+2(x-10) & \text{Factor by grouping} \\ (x-10)(x+2) & \text{Our Solution} \end{array}$$

Often when factoring we have two variables. These problems solve just like problems with one variable, using the coefficients to decide how to split the middle term

Example 6.

$$\begin{array}{rl} a^2-9ab+14b^2 & \text{Want to multiply to 14, add to} -9 \\ a^2-7ab-2ab+14b^2 & -7 \text{ and } -2\text{, split the middle term} \\ a(a-7b)-2b(a-7b) & \text{Factor by grouping} \\ (a-7b)(a-2b) & \text{Our Solution} \end{array}$$

As the past few examples illustrate, it is very important to be aware of negatives as we find the pair of numbers we will use to split the middle term. Consier the following example, done incorrectly, ignoring negative signs

Warning 7.

$$\begin{array}{rl} x^2+5x-6 & \text{Want to multiply to } 6, \text{add } 5 \\ x^2+2x+3x-6 & 2 \text{ and } 3, \text{split the middle term} \\ x(x+2)+3(x-2) & \text{Factor by grouping} \\ ??? & \text{Binomials do not match!} \end{array}$$

Because we did not use the negative sign with the six to find our pair of numbers, the binomials did not match and grouping was not able to work at the end. Now the problem will be done correctly.

Example 8.

$$\begin{array}{rl} x^2+5x-6 & \text{Want to multiply to } -6, \text{add to } 5 \\ x^2+6x-x-6 & 6 \text{ and } -1, \text{split the middle term} \\ x(x+6)-1(x+6) & \text{Factor by grouping} \\ (x+6)(x-1) & \text{Our Solution} \end{array}$$

You may have noticed a shortcut for factoring these problems. Once we identify the two numbers that are used to split the middle term, these are the two numbers in our factors! In the previous example, the numbers used to split the middle term were 6 and -1, our factors turned out to be $(x+6)(x-1)$. This pattern does not always work, so be careful getting in the habit of using it. We can use it however, when we have no number (technically we have a 1) in front of x^2. In all the problems we have factored in this lesson there is no number in front of x^2. If this is the case then we can use this shortcut. This is shown in the next few examples.

Example 9.

$$\begin{array}{rl} x^2-7x-18 & \text{Want to multiply to } -18, \text{add to } -7 \\ & -9 \text{ and } 2, \text{write the factors} \\ (x-9)(x+2) & \text{Our Solution} \end{array}$$

Example 10.

$$\begin{array}{rl} m^2 - mn - 30n^2 & \text{Want to multiply to} - 30\text{, add to} - 1 \\ & 5 \text{ and} - 6\text{, write the factors, don't forget second variable} \\ (m+5n)(m-6n) & \text{Our Solution} \end{array}$$

It is possible to have a problem that does not factor. If there is no combination of numbers that multiplies and adds to the correct numbers, then we say we cannot factor the polynomial, or we say the polynomial is prime. This is shown in the following example.

Example 11.

$$\begin{array}{rl} x^2 + 2x + 6 & \text{Want to multiply to 6, add to 2} \\ 1 \cdot 6 \text{ and } 2 \cdot 3 & \text{Only possibilities to multiply to six, none add to 2} \\ \text{Prime, can't factor} & \text{Our Solution} \end{array}$$

When factoring it is important not to forget about the GCF. If all the terms in a problem have a common factor we will want to first factor out the GCF before we factor using any other method.

Example 12.

$$\begin{array}{rl} 3x^2 - 24x + 45 & \text{GCF of all terms is 3, factor this out} \\ 3(x^2 - 8x + 15) & \text{Want to multiply to 15, add to} - 8 \\ & -5 \text{ and} - 3\text{, write the factors} \\ 3(x-5)(x-3) & \text{Our Solution} \end{array}$$

Again it is important to comment on the shortcut of jumping right to the factors, this only works if there is no coefficient on x^2. In the next lesson we will look at how this process changes slightly when we have a number in front of x^2. Be careful not to use this shortcut on all factoring problems!

World View Note: The first person to use letters for unknown values was Francois Vieta in 1591 in France. He used vowels to represent variables we are solving for, just as codes used letters to represent an unknown message.

3.3 Practice - Trinomials where a = 1

Factor each completely.

1) $p^2+17p+72$

2) x^2+x-72

3) n^2-9n+8

4) x^2+x-30

5) $x^2-9x-10$

6) $x^2+13x+40$

7) $b^2+12b+32$

8) $b^2-17b+70$

9) $x^2+3x-70$

10) $x^2+3x-18$

11) $n^2-8n+15$

12) $a^2-6a-27$

13) $p^2+15p+54$

14) $p^2+7p-30$

15) $n^2-15n+56$

16) $m^2-15mn+50n^2$

17) $u^2-8uv+15v^2$

18) $m^2-3mn-40n^2$

19) $m^2+2mn-8n^2$

20) $x^2+10xy+16y^2$

21) $x^2-11xy+18y^2$

22) $u^2-9uv+14v^2$

23) $x^2+xy-12y^2$

24) $x^2+14xy+45y^2$

25) $x^2+4xy-12y^2$

26) $4x^2+52x+168$

27) $5a^2+60a+100$

28) $5n^2-45n+40$

29) $6a^2+24a-192$

30) $5v^2+20v-25$

31) $6x^2+18xy+12y^2$

32) $5m^2+30mn-90n^2$

33) $6x^2+96xy+378y^2$

34) $6m^2-36mn-162n^2$

Answers - Trinomials where a = 1

1) $(p+9)(p+8)$
2) $(x-8)(x+9)$
3) $(n-8)(n-1)$
4) $(x-5)(x+6)$
5) $(x+1)(x-10)$
6) $(x+5)(x+8)$
7) $(b+8)(b+4)$
8) $(b-10)(b-7)$
9) $(x-7)(x+10)$
10) $(x-3)(x+6)$
11) $(n-5)(n-3)$
12) $(a+3)(a-9)$
13) $(p+6)(p+9)$
14) $(p+10)(p-3)$
15) $(n-8)(n-7)$
16) $(m-5n)(m-10n)$
17) $(u-5v)(u-3v)$
18) $(m+5n)(m-8n)$
19) $(m+4n)(m-2n)$
20) $(x+8y)(x+2y)$
21) $(x-9y)(x-2y)$
22) $(u-7v)(u-2v)$
23) $(x-3y)(x+4y)$
24) $(x+5y)(x+9y)$
25) $(x+6y)(x-2y)$
26) $4(x+7)(x+6)$
27) $5(a+10)(a+2)$
28) $5(n-8)(n-1)$
29) $6(a-4)(a+8)$
30) $5(v-1)(v+5)$
31) $6(x+2y)(x+y)$
32) $5(m^2+6mn-18n^2)$
33) $6(x+9y)(x+7y)$
34) $6(m-9n)(m+3n)$

3.4 Trinomials where a $\neq 1$

Objective: Factor trinomials using the ac method when the coefficient of x^2 is not one.

When factoring trinomials we used the ac method to split the middle term and then factor by grouping. The ac method gets it's name from the general trinomial equation, $ax^2 + bx + c$, where a, b, and c are the numbers in front of x^2, x and the constant at the end respectively.

World View Note: It was French philosopher Rene Descartes who first used letters from the beginning of the alphabet to represent values we know (a, b, c) and letters from the end to represent letters we don't know and are solving for (x, y, z).

The ac method is named ac because we multiply $a\cdot c$ to find out what we want to multiply to. In the previous lesson we always multiplied to just c because there was no number in front of x^2. This meant the number was 1 and we were multiplying to $1c$ or just c. Now we will have a number in front of x^2 so we will be looking for numbers that multiply to ac and add to b. Other than this, the process will be the same.

Example 1.

$$\begin{array}{rl} 3x^2+11x+6 & \text{Multiply to } ac \text{ or } (3)(6)=18, \text{add to } 11 \\ 3x^2+9x+2x+6 & \text{The numbers are 9 and 2, split the middle term} \\ 3x(x+3)+2(x+3) & \text{Factor by grouping} \\ (x+3)(3x+2) & \text{Our Solution} \end{array}$$

When $a=1$, or no coefficient in front of x^2, we were able to use a shortcut, using the numbers that split the middle term in the factors. The previous example illustrates an important point, the shortcut does not work when $a\neq 1$. We must go through all the steps of grouping in order to factor the problem.

Example 2.

$$\begin{array}{rl} 8x^2-2x-15 & \text{Multiply to } ac \text{ or } (8)(-15)=-120, \text{add to } -2 \\ 8x^2-12x+10x-15 & \text{The numbers are } -12 \text{ and } 10, \text{split the middle term} \\ 4x(2x-3)+5(2x-3) & \text{Factor by grouping} \\ (2x-3)(4x+5) & \text{Our Solution} \end{array}$$

Example 3.

$$\begin{aligned} 10x^2-27x+5 &\quad \text{Multiply to } ac \text{ or } (10)(5)=50, \text{ add to } -27\\ 10x^2-25x-2x+5 &\quad \text{The numbers are } -25 \text{ and } -2, \text{ split the middle term}\\ 5x(2x-5)-1(2x-5) &\quad \text{Factor by grouping}\\ (2x-5)(5x-1) &\quad \text{Our Solution}\end{aligned}$$

The same process works with two variables in the problem

Example 4.

$$\begin{aligned} 4x^2-xy-5y^2 &\quad \text{Multiply to } ac \text{ or } (4)(-5)=-20, \text{ add to } -1\\ 4x^2+4xy-5xy-5y^2 &\quad \text{The numbers are 4 and } -5, \text{ split the middle term}\\ 4x(x+y)-5y(x+y) &\quad \text{Factor by grouping}\\ (x+y)(4x-5y) &\quad \text{Our Solution}\end{aligned}$$

As always, when factoring we will first look for a GCF before using any other method, including the ac method. Factoring out the GCF first also has the added bonus of making the numbers smaller so the ac method becomes easier.

Example 5.

$$\begin{aligned} 18x^3+33x^2-30x &\quad \text{GCF} = 3x, \text{ factor this out first}\\ 3x[6x^2+11x-10] &\quad \text{Multiply to } ac \text{ or } (6)(-10)=-60, \text{ add to } 11\\ 3x[6x^2+15x-4x-10] &\quad \text{The numbers are 15 and } -4, \text{ split the middle term}\\ 3x[3x(2x+5)-2(2x+5)] &\quad \text{Factor by grouping}\\ 3x(2x+5)(3x-2) &\quad \text{Our Solution}\end{aligned}$$

As was the case with trinomials when $a=1$, not all trinomials can be factored. If there is no combinations that multiply and add correctly then we can say the trinomial is prime and cannot be factored.

Example 6.

$$\begin{aligned} 3x^2+2x-7 &\quad \text{Multiply to } ac \text{ or } (3)(-7)=-21, \text{ add to } 2\\ -3(7) \text{ and } -7(3) &\quad \text{Only two ways to multiply to } -21, \text{ it doesn't add to } 2\\ \text{Prime, cannot be factored} &\quad \text{Our Solution}\end{aligned}$$

3.4 Practice - Trinomials where a $\neq$ 1

Factor each completely.

1) $7x^2-48x+36$

2) $7n^2-44n+12$

3) $7b^2+15b+2$

4) $7v^2-24v-16$

5) $5a^2-13a-28$

6) $5n^2-4n-20$

7) $2x^2-5x+2$

8) $3r^2-4r-4$

9) $2x^2+19x+35$

10) $7x^2+29x-30$

11) $2b^2-b-3$

12) $5k^2-26k+24$

13) $5k^2+13k+6$

14) $3r^2+16r+21$

15) $3x^2-17x+20$

16) $3u^2+13uv-10v^2$

17) $3x^2+17xy+10y^2$

18) $7x^2-2xy-5y^2$

19) $5x^2+28xy-49y^2$

20) $5u^2+31uv-28v^2$

21) $6x^2-39x-21$

22) $10a^2-54a-36$

23) $21k^2-87k-90$

24) $21n^2+45n-54$

25) $14x^2-60x+16$

26) $4r^2+r-3$

27) $6x^2+29x+20$

28) $6p^2+11p-7$

29) $4k^2-17k+4$

30) $4r^2+3r-7$

31) $4x^2+9xy+2y^2$

32) $4m^2+6mn+6n^2$

33) $4m^2-9mn-9n^2$

34) $4x^2-6xy+30y^2$

35) $4x^2+13xy+3y^2$

36) $18u^2-3uv-36v^2$

37) $12x^2+62xy+70y^2$

38) $16x^2+60xy+36y^2$

39) $24x^2-52xy+8y^2$

40) $12x^2+50xy+28y^2$

Answers - Trinomials where a $\neq$ 1

1) $(7x-6)(x-6)$

2) $(7n-2)(n-6)$

3) $(7b+1)(b+2)$

4) $(7v+4)(v-4)$

5) $(5a+7)(a-4)$

6) Prime

7) $(2x-1)(x-2)$

8) $(3r+2)(r-2)$

9) $(2x+5)(x+7)$

10) $(7x-6)(x+5)$

11) $(2b-3)(b+1)$

12) $(5k-6)(k-4)$

13) $(5k+3)(k+2)$

14) $(3r+7)(r+3)$

15) $(3x-5)(x-4)$

16) $(3u-2v)(u+5v)$

17) $(3x+2y)(x+5y)$

18) $(7x+5y)(x-y)$

19) $(5x-7y)(x+7y)$

20) $(5u-4v)(u+7v)$

21) $3(2x+1)(x-7)$

22) $2(5a+3)(a-6)$

23) $3(7k+6)(k-5)$

24) $3(7n-6)(n+3)$

25) $2(7x-2)(x-4)$

26) $(r+1)(4r-3)$

27) $(x+4)(6x+5)$

28) $(3p+7)(2p-1)$

29) $(k-4)(4k-1)$

30) $(r-1)(4r+7)$

31) $(x+2y)(4x+y)$

32) $2(2m^2+3mn+3n^2)$

33) $(m-3n)(4m+3n)$

34) $2(2x^2-3xy+15y^2)$

35) $(x+3y)(4x+y)$

36) $3(3u+4v)(2u-3v)$

37) $2(2x+7y)(3x+5y)$

38) $4(x+3y)(4x+3y)$

39) $4(x-2y)(6x-y)$

40) $2(3x+2y)(2x+7y)$

3.5 Factoring Special Products

Objective: Identify and factor special products including a difference of squares, perfect squares, and sum and difference of cubes.

When factoring there are a few special products that, if we can recognize them, can help us factor polynomials. The first is one we have seen before. When multiplying special products we found that a sum and a difference could multiply to a difference of squares. Here we will use this special product to help us factor

Difference of Squares: $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$

If we are subtracting two perfect squares then it will always factor to the sum and difference of the square roots.

Example 1.

$x^2 - 16$	Subtracting two perfect squares, the square roots are x and 4
$(x+4)(x-4)$	Our Solution

Example 2.

$9a^2 - 25b^2$	Subtracting two perfect squares, the square roots are $3a$ and $5b$
$(3a+5b)(3a-5b)$	Our Solution

It is important to note, that a sum of squares will never factor. It is always prime. This can be seen if we try to use the ac method to factor $x^2 + 36$.

Example 3.

$x^2 + 36$	No bx term, we use $0x$.
$x^2 + 0x + 36$	Multiply to 36, add to 0
$1 \cdot 36, 2 \cdot 18, 3 \cdot 12, 4 \cdot 9, 6 \cdot 6$	No combinations that multiply to 36 add to 0
Prime, cannot factor	Our Solution

It turns out that a sum of squares is always prime.

Sum of Squares: $\boldsymbol{a^2 + b^2 = \text{Prime}}$

A great example where we see a sum of squares comes from factoring a difference of 4th powers. Because the square root of a fourth power is a square ($\sqrt{a^4} = a^2$), we can factor a difference of fourth powers just like we factor a difference of squares, to a sum and difference of the square roots. This will give us two factors, one which will be a prime sum of squares, and a second which will be a difference of squares which we can factor again. This is shown in the following examples.

Example 4.

$$\begin{array}{rl} a^4 - b^4 & \text{Difference of squares with roots } a^2 \text{ and } b^2 \\ (a^2 + b^2)(a^2 - b^2) & \text{The first factor is prime, the second is } a \text{ difference of squares!} \\ (a^2 + b^2)(a + b)(a - b) & \text{Our Solution} \end{array}$$

Example 5.

$$\begin{array}{rl} x^4 - 16 & \text{Difference of squares with roots } x^2 \text{ and } 4 \\ (x^2 + 4)(x^2 - 4) & \text{The first factor is prime, the second is } a \text{ difference of squares!} \\ (x^2 + 4)(x + 2)(x - 2) & \text{Our Solution} \end{array}$$

Another factoring shortcut is the perfect square. We had a shortcut for multiplying a perfect square which can be reversed to help us factor a perfect square

Perfect Square: $\boldsymbol{a^2 + 2\mathrm{ab} + b^2 = (a + b)^2}$

A perfect square can be difficult to recognize at first glance, but if we use the ac method and get two of the same numbers we know we have a perfect square. Then we can just factor using the square roots of the first and last terms and the sign from the middle. This is shown in the following examples.

Example 6.

$$\begin{array}{rl} x^2 - 6x + 9 & \text{Multiply to 9, add to} - 6 \\ & \text{The numbers are} - 3 \text{ and} - 3\text{, the same! Perfect square} \\ (x - 3)^2 & \text{Use square roots from first and last terms and sign from the middle} \end{array}$$

Example 7.

$$\begin{array}{rl} 4x^2 + 20xy + 25y^2 & \text{Multiply to 100, add to 20} \\ & \text{The numbers are 10 and 10, the same! Perfect square} \\ (2x + 5y)^2 & \text{Use square roots from first and last terms and sign from the middle} \end{array}$$

World View Note: The first known record of work with polynomials comes from the Chinese around 200 BC. Problems would be written as "three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop sold for 29 dou. This would be the polynomial (trinomial) $3x + 2y + z = 29$.

Another factoring shortcut has cubes. With cubes we can either do a sum or a difference of cubes. Both sum and difference of cubes have very similar factoring formulas

$$\textbf{Sum of Cubes: } \boldsymbol{a^3 + b^3 = (a + b)(a^2 - \mathrm{ab} + b^2)}$$

$$\textbf{Difference of Cubes: } \boldsymbol{a^3 - b^3 = (a - b)(a^2 + \mathrm{ab} + b^2)}$$

Comparing the formulas you may notice that the only difference is the signs in between the terms. One way to keep these two formulas straight is to think of SOAP. S stands for Same sign as the problem. If we have a sum of cubes, we add first, a difference of cubes we subtract first. O stands for Opposite sign. If we have a sum, then subtraction is the second sign, a difference would have addition for the second sign. Finally, AP stands for Always Positive. Both formulas end with addition. The following examples show factoring with cubes.

Example 8.

$$\begin{array}{rl} m^3 - 27 & \text{We have cube roots } m \text{ and } 3 \\ (m \quad 3)(m^2 \quad 3m \quad 9) & \text{Use formula, use SOAP to fill in signs} \\ (m - 3)(m^2 + 3m + 9) & \text{Our Solution} \end{array}$$

Example 9.

$$\begin{array}{rl} 125p^3 + 8r^3 & \text{We have cube roots } 5p \text{ and } 2r \\ (5p \quad 2r)(25p^2 \quad 10r \quad 4r^2) & \text{Use formula, use SOAP to fill in signs} \\ (5p + 2r)(25p^2 - 10r + 4r^2) & \text{Our Solution} \end{array}$$

The previous example illustrates an important point. When we fill in the trinomial's first and last terms we square the cube roots $5p$ and $2r$. Often students forget to square the number in addition to the variable. Notice that when done correctly, both get cubed.

Often after factoring a sum or difference of cubes, students want to factor the second factor, the trinomial further. As a general rule, this factor will always be prime (unless there is a GCF which should have been factored out before using cubes rule).

The following table sumarizes all of the shortcuts that we can use to factor special products

Factoring Special Products

Difference of Squares	$a^2 - b^2 = (a+b)(a-b)$
Sum of Squares	$a^2 + b^2 = \text{Prime}$
Perfect Square	$a^2 + 2ab + b^2 = (a+b)^2$
Sum of Cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of Cubes	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

As always, when factoring special products it is important to check for a GCF first. Only after checking for a GCF should we be using the special products. This is shown in the following examples

Example 10.

$72x^2 - 2$	GCF is 2
$2(36x^2 - 1)$	Difference of Squares, square roots are $6x$ and 1
$2(6x+1)(6x-1)$	Our Solution

Example 11.

$48x^2y - 24xy + 3y$	GCF is $3y$
$3y(16x^2 - 8x + 1)$	Multiply to 16 add to 8
	The numbers are 4 and 4, the same! Perfect Square
$3y(4x-1)^2$	Our Solution

Example 12.

$128a^4b^2 + 54ab^5$	GCF is $2ab^2$
$2ab^2(64a^3 + 27b^3)$	Sum of cubes! Cube roots are $4a$ and $3b$
$2\text{ab}^2(4a+3b)(16a^2 - 12ab + 9b^2)$	Our Solution

3.5 Practice - Factoring Special Products

Factor each completely.

1) $r^2 - 16$

2) $x^2 - 9$

3) $v^2 - 25$

4) $x^2 - 1$

5) $p^2 - 4$

6) $4v^2 - 1$

7) $9k^2 - 4$

8) $9a^2 - 1$

9) $3x^2 - 27$

10) $5n^2 - 20$

11) $16x^2 - 36$

12) $125x^2 + 45y^2$

13) $18a^2 - 50b^2$

14) $4m^2 + 64n^2$

15) $a^2 - 2a + 1$

16) $k^2 + 4k + 4$

17) $x^2 + 6x + 9$

18) $n^2 - 8n + 16$

19) $x^2 - 6x + 9$

20) $k^2 - 4k + 4$

21) $25p^2 - 10p + 1$

22) $x^2 + 2x + 1$

23) $25a^2 + 30ab + 9b^2$

24) $x^2 + 8xy + 16y^2$

25) $4a^2 - 20ab + 25b^2$

26) $18m^2 - 24mn + 8n^2$

27) $8x^2 - 24xy + 18y^2$

28) $20x^2 + 20xy + 5y^2$

29) $8 - m^3$

30) $x^3 + 64$

31) $x^3 - 64$

32) $x^3 + 8$

33) $216 - u^3$

34) $125x^3 - 216$

35) $125a^3 - 64$

36) $64x^3 - 27$

37) $64x^3 + 27y^3$

38) $32m^3 - 108n^3$

39) $54x^3 + 250y^3$

40) $375m^3 + 648n^3$

41) $a^4 - 81$

42) $x^4 - 256$

43) $16 - z^4$

44) $n^4 - 1$

45) $x^4 - y^4$

46) $16a^4 - b^4$

47) $m^4 - 81b^4$

48) $81c^4 - 16d^4$

Answers - Factoring Special Products

1) $(r+4)(r-4)$

2) $(x+3)(x-3)$

3) $(v+5)(v-5)$

4) $(x+1)(x-1)$

5) $(p+2)(p-2)$

6) $(2v+1)(2v-1)$

7) $(3k+2)(3k-2)$

8) $(3a+1)(3a-1)$

9) $3(x+3)(x-3)$

10) $5(n+2)(n-2)$

11) $4(2x+3)(2x-3)$

12) $5(25x^2+9y^2)$

13) $2(3a+5b)(3a-5b)$

14) $4(m^2+16n^2)$

15) $(a-1)^2$

16) $(k+2)^2$

17) $(x+3)^2$

18) $(n-4)^2$

19) $(x-3)^2$

20) $(k-2)^2$

21) $(5p-1)^2$

22) $(x+1)^2$

23) $(5a+3b)^2$

24) $(x+4y)^2$

25) $(2a-5b)^2$

26) $2(3m-2n)^2$

27) $2(2x-3y)^2$

28) $5(2x+y)^2$

29) $(2-m)(4+2m+m^2)$

30) $(x+4)(x^2-4x+16)$

31) $(x-4)(x^2+4x+16)$

32) $(x+2)(x^2-2x+4)$

33) $(6-u)(36+6u+u^2)$

34) $(5x-6)(25x^2+30x+36)$

35) $(5a-4)(25a^2+20a+16)$

36) $(4x-3)(16x^2+12x+9)$

37) $(4x+3y)(16x^2-12xy+9y^2)$

38) $4(2m-3n)(4m^2+6mn+9n^2)$

39) $2(3x+5y)(9x^2-15xy+25y^2)$

40) $3(5m+6n)(25m^2-30mn+36n^2)$

41) $(a^2+9)(a+3)(a-3)$

42) $(x^2+16)(x+4)(x-4)$

43) $(4+z^2)(2+z)(2-z)$

44) $(n^2+1)(n+1)(n-1)$

45) $(x^2+y^2)(x+y)(x-y)$

46) $(4a^2+b^2)(2a+b)(2a-b)$

47) $(m^2+9b^2)(m+3b)(m-3b)$

48) $(9c^2+4d^2)(3c+2d)(3c-2d)$

3.6 Factoring Strategy

Objective: Idenfity and use the correct method to factor various polynomials.

With so many different tools used to factor, it is easy to get lost as to which tool to use when. Here we will attempt to organize all the different factoring types we have seen. A large part of deciding how to solve a problem is based on how many terms are in the problem. For all problem types we will always try to factor out the GCF first.

Factoring Strategy (GCF First!!!!!)

- **2 terms:** sum or difference of squares or cubes:

  $a^2 - b^2 = (a+b)(a-b)$

  $a^2 + b^2 = \text{Prime}$

  $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

  $a^3 - b^3 = (a-b)(a^2 + \text{ab} + b^2)$

- **3 terms:** ac method, watch for perfect square!

  $a^2 + 2ab + b^2 = (a+b)^2$

  Multiply to ac and add to b

- **4 terms:** grouping

We will use the above strategy to factor each of the following examples. Here the emphasis will be on which strategy to use rather than the steps used in that method.

Example 1.

$$\begin{array}{rl} 4x^2 + 56xy + 196y^2 & \text{GCF first, 4} \\ 4(x^2 + 14xy + 49y^2) & \text{Three terms, try ac method, multiply to 49, add to 14} \\ & \text{7 and 7, perfect square!} \\ 4(x+7y)^2 & \text{Our Solution} \end{array}$$

Example 2.

$$
\begin{aligned}
5x^2y + 15xy - 35x^2 - 105x &\quad \text{GCF first, } 5x \\
5x(xy + 3y - 7x - 21) &\quad \text{Four terms, try grouping} \\
5x[y(x+3) - 7(x+3)] &\quad (x+3) \text{ match!} \\
5x(x+3)(y-7) &\quad \text{Our Solution}
\end{aligned}
$$

Example 3.

$$
\begin{aligned}
100x^2 - 400 &\quad \text{GCF first, } 100 \\
100(x^2 - 4) &\quad \text{Two terms, difference of squares} \\
100(x+4)(x-4) &\quad \text{Our Solution}
\end{aligned}
$$

Example 4.

$$
\begin{aligned}
108x^3y^2 - 39x^2y^2 + 3xy^2 &\quad \text{GCF first, } 3xy^2 \\
3xy^2(36x^2 - 13x + 1) &\quad \text{Thee terms, ac method, multiply to 36, add to } -13 \\
3xy^2(36x^2 - 9x - 4x + 1) &\quad -9 \text{ and } -4\text{, split middle term} \\
3xy^2[9x(4x-1) - 1(4x-1)] &\quad \text{Factor by grouping} \\
3xy^2(4x-1)(9x-1) &\quad \text{Our Solution}
\end{aligned}
$$

World View Note: Variables originated in ancient Greece where Aristotle would use a single capital letter to represent a number.

Example 5.

$$
\begin{aligned}
5 + 625y^3 &\quad \text{GCF first, } 5 \\
5(1 + 125y^3) &\quad \text{Two terms, sum of cubes} \\
5(1+5y)(1 - 5y + 25y^2) &\quad \text{Our Solution}
\end{aligned}
$$

It is important to be comfortable and confident not just with using all the factoring methods, but decided on which method to use. This is why practice is very important!

3.6 Practice - Factoring Strategy

Factor each completely.

1) $24az - 18ah + 60yz - 45yh$

2) $2x^2 - 11x + 15$

3) $5u^2 - 9uv + 4v^2$

4) $16x^2 + 48xy + 36y^2$

5) $-2x^3 + 128y^3$

6) $20uv - 60u^3 - 5xv + 15xu^2$

7) $5n^3 + 7n^2 - 6n$

8) $2x^3 + 5x^2y + 3y^2x$

9) $54u^3 - 16$

10) $54 - 128x^3$

11) $n^2 - n$

12) $5x^2 - 22x - 15$

13) $x^2 - 4xy + 3y^2$

14) $45u^2 - 150uv + 125v^2$

15) $9x^2 - 25y^2$

16) $x^3 - 27y^3$

17) $m^2 - 4n^2$

18) $12ab - 18a + 6nb - 9n$

19) $36b^2c - 16xd - 24b^2d + 24xc$

20) $3m^3 - 6m^2n - 24n^2m$

21) $128 + 54x^3$

22) $64m^3 + 27n^3$

23) $2x^3 + 6x^2y - 20y^2x$

24) $3ac + 15ad^2 + x^2c + 5x^2d^2$

25) $n^3 + 7n^2 + 10n$

26) $64m^3 - n^3$

27) $27x^3 - 64$

28) $16a^2 - 9b^2$

29) $5x^2 + 2x$

30) $2x^2 - 10x + 12$

31) $3k^3 - 27k^2 + 60k$

32) $32x^2 - 18y^2$

33) $mn - 12x + 3m - 4xn$

34) $2k^2 + k - 10$

35) $16x^2 - 8\text{xy} + y^2$

36) $v^2 + v$

37) $27m^2 - 48n^2$

38) $x^3 + 4x^2$

39) $9x^3 + 21x^2y - 60y^2x$

40) $9n^3 - 3n^2$

41) $2m^2 + 6mn - 20n^2$

42) $2u^2v^2 - 11uv^3 + 15v^4$

Answers - Factoring Strategy

1) $3(2a+5y)(4z-3h)$

2) $(2x-5)(x-3)$

3) $(5u-4v)(u-v)$

4) $4(2x+3y)^2$

5) $2(-x+4y)(x^2+4xy+16y^2)$

6) $5(4u-x)(v-3u^2)$

7) $n(5n-3)(n+2)$

8) $x(2x+3y)(x+y)$

9) $2(3u-2)(9u^2+6u+4)$

10) $2(3-4x)(9+12x+16x^2)$

11) $n(n-1)$

12) $(5x+3)(x-5)$

13) $(x-3y)(x-y)$

14) $5(3u-5v)^2$

15) $(3x+5y)(3x-5y)$

16) $(x-3y)(x^2+3xy+9y^2)$

17) $(m+2n)(m-2n)$

18) $3(2a+n)(2b-3)$

19) $4(3b^2+2x)(3c-2d)$

20) $3m(m+2n)(m-4n)$

21) $2(4+3x)(16-12x+9x^2)$

22) $(4m+3n)(16m^2-12mn+9n^2)$

23) $2x(x+5y)(x-2y)$

24) $(3a+x^2)(c+5d^2)$

25) $n(n+2)(n+5)$

26) $(4m-n)(16m^2+4mn+n^2)$

27) $(3x-4)(9x^2+12x+16)$

28) $(4a+3b)(4a-3b)$

29) $x(5x+2)$

30) $2(x-2)(x-3)$

31) $3k(k-5)(k-4)$

32) $2(4x+3y)(4x-3y)$

33) $(m-4x)(n+3)$

34) $(2k+5)(k-2)$

35) $(4x-y)^2$

36) $v(v+1)$

37) $3(3m+4n)(3m-4n)$

38) $x^2(x+4)$

39) $3x(3x-5y)(x+4y)$

40) $3n^2(3n-1)$

41) $2(m-2n)(m+5n)$

42) $v^2(2u-5v)(u-3v)$

3.7 Solve by Factoring

Objective: Solve quadratic equation by factoring and using the zero product rule.

When solving linear equations such as $2x-5=21$ we can solve for the variable directly by adding 5 and dividing by 2 to get 13. However, when we have x^2 (or a higher power of x) we cannot just isolate the variable as we did with the linear equations. One method that we can use to solve for the varaible is known as the zero product rule

Zero Product Rule: If $ab=0$ then either $a=0$ or $b=0$

The zero product rule tells us that if two factors are multiplied together and the answer is zero, then one of the factors must be zero. We can use this to help us solve factored polynomials as in the following example.

Example 1.

$$\begin{array}{rl}
(2x-3)(5x+1)=0 & \text{One factor must be zero}\\
2x-3=0 \text{ or } 5x+1=0 & \text{Set each factor equal to zero}\\
\underline{+3+3} \qquad \underline{-1-1} & \text{Solve each equation}\\
2x=3 \text{ or } 5x=-1 & \\
\overline{2}\ \overline{2} \qquad \overline{5}\ \ \overline{5} & \\
x=\dfrac{3}{2} \text{ or } \dfrac{-1}{5} & \text{Our Solution}
\end{array}$$

For the zero product rule to work we must have factors to set equal to zero. This means if the problem is not already factored we will factor it first.

Example 2.

$$\begin{array}{rl}
4x^2+x-3=0 & \text{Factor using the ac method, multiply to } -12\text{, add to } 1\\
4x^2-3x+4x-3=0 & \text{The numbers are } -3 \text{ and } 4\text{, split the middle term}\\
x(4x-3)+1(4x-3)=0 & \text{Factor by grouping}\\
(4x-3)(x+1)=0 & \text{One factor must be zero}\\
4x-3=0 \text{ or } x+1=0 & \text{Set each factor equal to zero}\\
\underline{+3+3} \qquad \underline{-1-1} & \text{Solve each equation}\\
4x=3 \text{ or } x=-1 & \\
\overline{4}\ \overline{4} \qquad\qquad & \\
x=\dfrac{3}{4} \text{ or } -1 & \text{Our Solution}
\end{array}$$

Another important part of the zero product rule is that before we factor, the equation must equal zero. If it does not, we must move terms around so it does equal zero. Generally we like the x^2 term to be positive.

Example 3.

$$\begin{array}{rl}
x^2 = 8x - 15 & \text{Set equal to zero by moving terms to the left} \\
\underline{-8x+15} \quad \underline{-8x+15} & \\
x^2 - 8x + 15 = 0 & \text{Factor using the ac method, multiply to } 15\text{, add to } -8 \\
(x-5)(x-3) = 0 & \text{The numbers are } -5 \text{ and } -3 \\
x - 5 = 0 \ \text{ or } \ x - 3 = 0 & \text{Set each factor equal to zero} \\
\underline{+5+5} \qquad \underline{+3+3} & \text{Solve each equation} \\
x = 5 \ \text{ or } \ x = 3 & \text{Our Solution}
\end{array}$$

Example 4.

$$\begin{array}{rl}
(x-7)(x+3) = -9 & \text{Not equal to zero, multiply first, use FOIL} \\
x^2 - 7x + 3x - 21 = -9 & \text{Combine like terms} \\
x^2 - 4x - 21 = -9 & \text{Move } -9 \text{ to other side so equation equals zero} \\
\underline{+9 \quad +9} & \\
x^2 - 4x - 12 = 0 & \text{Factor using the ac method, mutiply to } -12\text{, add to } -4 \\
(x-6)(x+2) = 0 & \text{The numbers are } 6 \text{ and } -2 \\
x - 6 = 0 \ \text{ or } \ x + 2 = 0 & \text{Set each factor equal to zero} \\
\underline{+6+6} \qquad \underline{-2-2} & \text{Solve each equation} \\
x = 6 \ \text{ or } \ -2 & \text{Our Solution}
\end{array}$$

Example 5.

$$\begin{array}{rl}
3x^2 + 4x - 5 = 7x^2 + 4x - 14 & \text{Set equal to zero by moving terms to the right} \\
\underline{-3x^2 - 4x + 5} \quad \underline{-3x^2 - 4x + 5} & \\
0 = 4x^2 - 9 & \text{Factor using difference of squares} \\
0 = (2x+3)(2x-3) & \text{One factor must be zero} \\
2x + 3 = 0 \ \text{ or } \ 2x - 3 = 0 & \text{Set each factor equal to zero} \\
\underline{-3-3} \qquad \underline{+3+3} & \text{Solve each equation} \\
2x = -3 \ \text{ or } \ 2x = 3 & \\
\overline{2} \quad \overline{2} \qquad \overline{2} \quad \overline{2} & \\
x = \dfrac{-3}{2} \ \text{ or } \ \dfrac{3}{2} & \text{Our Solution}
\end{array}$$

Most problems with x^2 will have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

Example 6.

$$
\begin{array}{rl}
4x^2 = 12x - 9 & \text{Set equal to zero by moving terms to left} \\
\underline{-12x + 9 \quad -12x + 9} & \\
4x^2 - 12x + 9 = 0 & \text{Factor using the ac method, multiply to 36, add to} - 12 \\
(2x - 3)^2 = 0 & -6 \text{ and} -6, a \text{ perfect square!} \\
2x - 3 = 0 & \text{Set this factor equal to zero} \\
\underline{+3 + 3} & \text{Solve the equation} \\
2x = 3 & \\
\overline{2} \quad \overline{2} & \\
x = \dfrac{3}{2} & \text{Our Solution}
\end{array}
$$

As always it will be important to factor out the GCF first if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This may give us more than just two solution. The next few examples illustrate this.

Example 7.

$$
\begin{array}{rl}
4x^2 = 8x & \text{Set equal to zero by moving the terms to left} \\
-8x - 8x & \text{Be careful, on the right side, they are not like terms!} \\
4x^2 - 8x = 0 & \text{Factor out the GCF of } 4x \\
4x(x - 2) = 0 & \text{One factor must be zero} \\
4x = 0 \text{ or } x - 2 = 0 & \text{Set each factor equal to zero} \\
\overline{4} \quad \overline{4} \qquad \underline{+2 + 2} & \text{Solve each equation} \\
x = 0 \text{ or } 2 & \text{Our Solution}
\end{array}
$$

Example 8.

$$
\begin{array}{rl}
2x^3 - 14x^2 + 24x = 0 & \text{Factor out the GCF of } 2x \\
2x(x^2 - 7x + 12) = 0 & \text{Factor with ac method, multiply to 12, add to} - 7 \\
2x(x - 3)(x - 4) = 0 & \text{The numbers are} - 3 \text{ and} - 4 \\
2x = 0 \text{ or } x - 3 = 0 \text{ or } x - 4 = 0 & \text{Set each factor equal to zero} \\
\overline{2} \quad \overline{2} \qquad \underline{+3 + 3} \qquad \underline{+4 + 4} & \text{Solve each equation} \\
x = 0 \text{ or } 3 \text{ or } 4 & \text{Our Solutions}
\end{array}
$$

Example 9.

$$\begin{array}{rl}
6x^2+21x-27=0 & \text{Factor out the GCF of 3} \\
3(2x^2+7x-9)=0 & \text{Factor with ac method, multiply to} -18\text{, add to 7} \\
3(2x^2+9x-2x-9)=0 & \text{The numbers are 9 and} -2 \\
3[x(2x+9)-1(2x+9)]=0 & \text{Factor by grouping} \\
3(2x+9)(x-1)=0 & \text{One factor must be zero} \\
3=0 \ \text{ or } \ 2x+9=0 \ \text{ or } \ x-1=0 & \text{Set each factor equal to zero} \\
3\neq 0 \qquad \underline{-9-9} \qquad \underline{+1+1} & \text{Solve each equation} \\
2x=-9 \ \text{ or } \ x=1 & \\
\overline{2} \quad \overline{2} \qquad\qquad & \\
x=-\dfrac{9}{2} \ \text{ or } \ 1 & \text{Our Solution}
\end{array}$$

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we got a false statement. No solutions come from this factor. Often a student will skip setting the GCF factor equal to zero if there is no variables in the GCF.

Just as not all polynomials cannot factor, all equations cannot be solved by factoring. If an equation does not factor we will have to solve it using another method. These other methods are saved for another section.

World View Note: While factoring works great to solve problems with x^2, Tartaglia, in 16th century Italy, developed a method to solve problems with x^3. He kept his method a secret until another mathematician, Cardan, talked him out of his secret and published the results. To this day the formula is known as Cardan's Formula.

A question often asked is if it is possible to get rid of the square on the variable by taking the square root of both sides. While it is possible, there are a few properties of square roots that we have not covered yet and thus it is common to break a rule of roots that we are not aware of at this point. The short reason we want to avoid this for now is because taking a square root will only give us one of the two answers. When we talk about roots we will come back to problems like these and see how we can solve using square roots in a method called completing the square. For now, **never** take the square root of both sides!

3.7 Practice - Solve by Factoring

Solve each equation by factoring.

1) $(k-7)(k+2)=0$

2) $(a+4)(a-3)=0$

3) $(x-1)(x+4)=0$

4) $(2x+5)(x-7)=0$

5) $6x^2-150=0$

6) $p^2+4p-32=0$

7) $2n^2+10n-28=0$

8) $m^2-m-30=0$

9) $7x^2+26x+15=0$

10) $40r^2-285r-280=0$

11) $5n^2-9n-2=0$

12) $2b^2-3b-2=0$

13) $x^2-4x-8=-8$

14) $v^2-8v-3=-3$

15) $x^2-5x-1=-5$

16) $a^2-6a+6=-2$

17) $49p^2+371p-163=5$

18) $7k^2+57k+13=5$

19) $7x^2+17x-20=-8$

20) $4n^2-13n+8=5$

21) $7r^2+84=-49r$

22) $7m^2-224=28m$

23) $x^2-6x=16$

24) $7n^2-28n=0$

25) $3v^2+7v=40$

26) $6b^2=5+7b$

27) $35x^2+120x=-45$

28) $9n^2+39n=-36$

29) $4k^2+18k-23=6k-7$

30) $a^2+7a-9=-3+6a$

31) $9x^2-46+7x=7x+8x^2+3$

32) $x^2+10x+30=6$

33) $2m^2+19m+40=-2m$

34) $5n^2+41n+40=-2$

35) $40p^2+183p-168=p+5p^2$

36) $24x^2+11x-80=3x$

Answers - Solve by Factoring

1) $7, -2$

2) $-4, 3$

3) $1, -4$

4) $-\frac{5}{2}, 7$

5) $-5, 5$

6) $4, -8$

7) $2, -7$

8) $-5, 6$

9) $-\frac{5}{7}, -3$

10) $-\frac{7}{8}, 8$

11) $-\frac{1}{5}, 2$

12) $-\frac{1}{2}, 2$

13) $4, 0$

14) $8, 0$

15) $1, 4$

16) $4, 2$

17) $\frac{3}{7}, -8$

18) $-\frac{1}{7}, -8$

19) $\frac{4}{7}, -3$

20) $\frac{1}{4}, 3$

21) $-4, -3$

22) $8, -4$

23) $8, -2$

24) $4, 0$

25) $\frac{8}{3}, -5$

26) $-\frac{1}{2}, \frac{5}{3}$

27) $-\frac{3}{7}, -3$

28) $-\frac{4}{3}, -3$

29) $-4, 1$

30) $2, -3$

31) $-7, 7$

32) $-4, -6$

33) $-\frac{5}{2}, -8$

34) $-\frac{6}{5}, -7$

35) $\frac{4}{5}, -6$

36) $\frac{5}{3}, -2$

Chapter 4 - Rational Expressions
4.1 Reduce Rational Expressions

Objective: Reduce rational expressions by dividing out common factors.

Rational expressions are expressions written as a quotient of polynomials. Examples of rational expressions include:

$$\frac{x^2-x-12}{x^2-9x+20} \quad \text{and} \quad \frac{3}{x-2} \quad \text{and} \quad \frac{a-b}{b-a} \quad \text{and} \quad \frac{3}{2}$$

As rational expressions are a special type of fraction, it is important to remember with fractions we cannot have zero in the denominator of a fraction. For this reason, rational expressions may have one more excluded values, or values that the variable cannot be or the expression would be undefined.

Example 1.

$$\begin{array}{rl}
\text{State the excluded value}(s)\text{:}\ \dfrac{x^2-1}{3x^2+5x} & \text{Denominator cannot be zero} \\
3x^2+5x\neq 0 & \text{Factor} \\
x(3x+5)\neq 0 & \text{Set each factor not equal to zero} \\
x\neq 0 \ \text{ or } \ 3x+5\neq 0 & \text{Subtract 5 from second equation} \\
\underline{-5-5} & \\
3x\neq -5 & \text{Divide by 3} \\
\overline{3}\quad\ \overline{3} & \\
x\neq \dfrac{-5}{3} & \text{Second equation is solved} \\
x\neq 0 \ \text{ or } \ \dfrac{-5}{3} & \text{Our Solution}
\end{array}$$

This means we can use any value for x in the equation except for 0 and $\frac{-5}{3}$. We can however, evaluate any other value in the expression.

World View Note: The number zero was not widely accepted in mathematical thought around the world for many years. It was the Mayans of Central America who first used zero to aid in the use of their base 20 system as a place holder!

Rational expressions are easily evaluated by simply substituting the value for the variable and using order of operations.

Example 2.

$$\frac{x^2-4}{x^2+6x+8} \text{ when } x=-6 \quad \text{Substitute } -5 \text{ in for each variable}$$

$$\frac{(-6)^2-4}{(-6)^2+6(-6)+8} \quad \text{Exponents first}$$

$$\frac{36-4}{36+6(-6)+8} \quad \text{Multiply}$$

$$\frac{36-4}{36-36+8} \quad \text{Add and subtract}$$

$$\frac{32}{8} \quad \text{Reduce}$$

$$4 \quad \text{Our Solution}$$

Just as we reduced the previous example, often a rational expression can be reduced, even without knowing the value of the variable. When we reduce we divide out common factors. We have already seen this with monomials when we discussed properties of exponents. If the problem only has monomials we can reduce the coefficients, and subtract exponents on the variables.

Example 3.

$$\frac{15x^4y^2}{25x^2y^6} \quad \text{Reduce, subtract exponents. Negative exponents move to denominator}$$

$$\frac{3x^2}{5y^4} \quad \text{Our Solution}$$

However, if there is more than just one term in either the numerator or denominator, we can't divide out common factors unless we first factor the numerator and denominator.

Example 4.

$$\frac{28}{8x^2-16} \quad \text{Denominator has } a \text{ common factor of } 8$$

$$\frac{28}{8(x^2-2)} \quad \text{Reduce by dividing 24 and 8 by 4}$$

$$\frac{7}{2(x^2-2)} \quad \text{Our Solution}$$

Example 5.

$$\frac{9x-3}{18x-6} \quad \text{Numerator has } a \text{ common factor of } 3, \text{ denominator of } 6$$

$$\frac{3(3x-1)}{6(3x-1)} \quad \text{Divide out common factor } (3x-1) \text{ and divide 3 and 6 by 3}$$

$$\frac{1}{2} \quad \text{Our Solution}$$

Example 6.

$$\frac{x^2-25}{x^2+8x+15} \quad \text{Numerator is difference of squares, denominator is factored using ac}$$

$$\frac{(x+5)(x-5)}{(x+3)(x+5)} \quad \text{Divide out common factor } (x+5)$$

$$\frac{x-5}{x+3} \quad \text{Our Solution}$$

It is important to remember we cannot reduce terms, only factors. This means if there are any $+$ or $-$ between the parts we want to reduce we cannot. In the previous example we had the solution $\frac{x-5}{x+3}$, we cannot divide out the x's because they are terms (separated by $+$ or $-$) not factors (separated by multiplication).

4.1 Practice - Reduce Rational Expressions

Evaluate

1) $\frac{4v+2}{6}$ when $v=4$

2) $\frac{b-3}{3b-9}$ when $b=-2$

3) $\frac{x-3}{x^2-4x+3}$ when $x=-4$

4) $\frac{a+2}{a^2+3a+2}$ when $a=-1$

5) $\frac{b+2}{b^2+4b+4}$ when $b=0$

6) $\frac{n^2-n-6}{n-3}$ when $n=4$

State the excluded values for each.

7) $\frac{3k^2+30k}{k+10}$

8) $\frac{27p}{18p^2-36p}$

9) $\frac{15n^2}{10n+25}$

10) $\frac{x+10}{8x^2+80x}$

11) $\frac{10m^2+8m}{10m}$

12) $\frac{10x+16}{6x+20}$

13) $\frac{r^2+3r+2}{5r+10}$

14) $\frac{6n^2-21n}{6n^2+3n}$

15) $\frac{b^2+12b+32}{b^2+4b-32}$

16) $\frac{10v^2+30v}{35v^2-5v}$

Simplify each expression.

17) $\frac{21x^2}{18x}$

18) $\frac{12n}{4n^2}$

19) $\frac{24a}{40a^2}$

20) $\frac{21k}{24k^2}$

21) $\frac{32x^3}{8x^4}$

22) $\frac{90x^2}{20x}$

23) $\frac{18m-24}{60}$

24) $\frac{10}{81n^3+36n^2}$

25) $\frac{20}{4p+2}$

26) $\frac{n-9}{9n-81}$

27) $\frac{x+1}{x^2+8x+7}$

28) $\frac{28m+12}{36}$

29) $\frac{32x^2}{28x^2+28x}$

30) $\frac{49r+56}{56r}$

31) $\frac{n^2+4n-12}{n^2-7n+10}$

32) $\frac{b^2+14b+48}{b^2+15b+56}$

33) $\frac{9v+54}{v^2-4v-60}$

34) $\frac{30x-90}{50x+40}$

35) $\frac{12x^2-42x}{30x^2-42x}$

36) $\frac{k^2-12k+32}{k^2-64}$

37) $\frac{6a-10}{10a+4}$

38) $\frac{9p+18}{p^2+4p+4}$

39) $\frac{2n^2+19n-10}{9n+90}$

40) $\frac{3x^2-29x+40}{5x^2-30x-80}$

41) $\frac{8m+16}{20m-12}$

42) $\frac{56x-48}{24x^2+56x+32}$

43) $\frac{2x^2-10x+8}{3x^2-7x+4}$

44) $\frac{50b-80}{50b+20}$

45) $\frac{7n^2-32n+16}{4n-16}$

46) $\frac{35v+35}{21v+7}$

47) $\frac{n^2-2n+1}{6n+6}$

48) $\frac{56x-48}{24x^2+56x+32}$

49) $\frac{7a^2-26a-45}{6a^2-34a+20}$

50) $\frac{4k^3-2k^2-2k}{9k^3-18k^2+9k}$

Answers - Reduce Rational Expressions

1) 3

2) $\frac{1}{3}$

3) $-\frac{1}{5}$

4) undefined

5) $\frac{1}{2}$

6) 6

7) -10

8) $0, 2$

9) $-\frac{5}{2}$

10) $0, -10$

11) 0

12) $-\frac{10}{3}$

13) -2

14) $0, -\frac{1}{2}$

15) $-8, 4$

16) $0, \frac{1}{7}$

17) $\frac{7x}{6}$

18) $\frac{3}{n}$

19) $\frac{3}{5a}$

20) $\frac{7}{8k}$

21) $\frac{4}{x}$

22) $\frac{9x}{2}$

23) $\frac{3m-4}{10}$

24) $\frac{10}{9n^2(9n+4)}$

25) $\frac{10}{2p+1}$

26) $\frac{1}{9}$

27) $\frac{1}{x+7}$

28) $\frac{7m+3}{9}$

29) $\frac{8x}{7(x+1)}$

30) $\frac{7r+8}{8r}$

31) $\frac{n+6}{n-5}$

32) $\frac{b+6}{b+7}$

33) $\frac{9}{v-10}$

34) $\frac{3(x-3)}{5x+4}$

35) $\frac{2x-7}{5x-7}$

36) $\frac{k-8}{k+4}$

37) $\frac{3a-5}{5a+2}$

38) $\frac{9}{p+2}$

39) $\frac{2n-1}{9}$

40) $\frac{3x-5}{5(x+2)}$

41) $\frac{2(m+2)}{5m-3}$

42) $\frac{9r}{5(r+1)}$

43) $\frac{2(x-4)}{3x-4}$

44) $\frac{5b-8}{5b+2}$

45) $\frac{7n-4}{4}$

46) $\frac{5(v+1)}{3v+1}$

47) $\frac{(n-1)^2}{6(n+1)}$

48) $\frac{7x-6}{(3x+4)(x+1)}$

49) $\frac{7a+9}{2(3a-2)}$

50) $\frac{2(2k+1)}{9(k-1)}$

4.2 Multiply & Divide

Objective: Multiply and divide rational expressions.

Multiplying and dividing rational expressions is very similar to the process we use to multiply and divide fractions.

Example 1.

$\frac{15}{49}\cdot\frac{14}{45}$	First reduce common factors from numerator and denominator (15 and 7)
$\frac{1}{7}\cdot\frac{2}{3}$	Multiply numerators across and denominators across
$\frac{2}{21}$	Our Solution

The process is identical for division with the extra first step of multiplying by the reciprocal. When multiplying with rational expressions we follow the same process, first divide out common factors, then multiply straight across.

Example 2.

$\frac{25x^2}{9y^8}\cdot\frac{24y^4}{55x^7}$	Reduce coefficients by dividing out common factors (3 and 5) Reduce, subtracting exponents, negative exponents in denominator
$\frac{5}{3y^4}\cdot\frac{8}{11x^5}$	Multiply across
$\frac{40}{33x^5y^4}$	Our Solution

Division is identical in process with the extra first step of multiplying by the reciprocal.

Example 3.

$$\frac{a^4b^2}{a} \div \frac{b^4}{4} \quad \text{Multiply by the reciprocal}$$

$$\frac{a^4b^2}{a} \cdot \frac{4}{b^4} \quad \text{Subtract exponents on variables, negative exponents in denominator}$$

$$\frac{a^3}{1} \cdot \frac{4}{b^2} \quad \text{Multiply across}$$

$$\frac{4a^3}{b^2} \quad \text{Our Solution}$$

Just as with reducing rational expressions, before we reduce a multiplication problem, it must be factored first.

Example 4.

$$\frac{x^2-9}{x^2+x-20} \cdot \frac{x^2-8x+16}{3x+9} \quad \text{Factor each numerator and denominator}$$

$$\frac{(x+3)(x-3)}{(x-4)(x+5)} \cdot \frac{(x-4)(x-4)}{3(x+3)} \quad \text{Divide out common factors } (x+3) \text{ and } (x-4)$$

$$\frac{x-3}{x+5} \cdot \frac{x-4}{3} \quad \text{Multiply across}$$

$$\frac{(x-3)(x-4)}{3(x+5)} \quad \text{Our Solution}$$

Again we follow the same pattern with division with the extra first step of multiplying by the reciprocal.

Example 5.

$$\frac{x^2-x-12}{x^2-2x-8} \div \frac{5x^2+15x}{x^2+x-2} \quad \text{Multiply by the reciprocal}$$

$$\frac{x^2-x-12}{x^2-2x-8} \cdot \frac{x^2+x-2}{5x^2+15x} \quad \text{Factor each numerator and denominator}$$

$$\frac{(x-4)(x+3)}{(x+2)(x-4)} \cdot \frac{(x+2)(x-1)}{5x(x+3)} \quad \text{Divide out common factors:}$$

$(x-4)$ and $(x+3)$ and $(x+2)$

$$\frac{1}{1} \cdot \frac{x-1}{5x} \quad \text{Multiply across}$$

$$\frac{x-1}{5x} \quad \text{Our Solution}$$

We can combine multiplying and dividing of fractions into one problem as shown below. To solve we still need to factor, and we use the reciprocal of the divided fraction.

Example 6.

$$\frac{a^2+7a+10}{a^2+6a+5} \cdot \frac{a+1}{a^2+4a+4} \div \frac{a-1}{a+2} \quad \text{Factor each expression}$$

$$\frac{(a+5)(a+2)}{(a+5)(a+1)} \cdot \frac{(a+1)}{(a+2)(a+2)} \div \frac{(a-1)}{(a+2)} \quad \text{Reciprocal of last fraction}$$

$$\frac{(a+5)(a+2)}{(a+5)(a+1)} \cdot \frac{(a+1)}{(a+2)(a+2)} \cdot \frac{(a+2)}{(a-1)} \quad \text{Divide out common factors}$$

$(a+2), (a+2), (a+1), (a+5)$

$$\frac{1}{a-1} \quad \text{Our Solution}$$

World View Note: Indian mathematician Aryabhata, in the 6th century, published a work which included the rational expression $\frac{n(n+1)(n+2)}{6}$ for the sum of the first n squares $(1^1+2^2+3^2+....+n^2)$

4.2 Practice - Multiply and Divide

Simplify each expression.

1) $\frac{8x^2}{9}\cdot\frac{9}{2}$

2) $\frac{8x}{3x}\div\frac{4}{7}$

3) $\frac{9n}{2n}\cdot\frac{7}{5n}$

4) $\frac{9m}{5m^2}\cdot\frac{7}{2}$

5) $\frac{5x^2}{4}\cdot\frac{6}{5}$

6) $\frac{10p}{5}\div\frac{8}{10}$

7) $\frac{7(m-6)}{m-6}\cdot\frac{5m(7m-5)}{7(7m-5)}$

8) $\frac{7}{10(n+3)}\div\frac{n-2}{(n+3)(n-2)}$

9) $\frac{7r}{7r(r+10)}\div\frac{r-6}{(r-6)^2}$

10) $\frac{6x(x+4)}{x-3}\cdot\frac{(x-3)(x-6)}{6x(x-6)}$

11) $\frac{25n+25}{5}\cdot\frac{4}{30n+30}$

12) $\frac{9}{b^2-b-12}\div\frac{b-5}{b^2-b-12}$

13) $\frac{x-10}{35x+21}\div\frac{7}{35x+21}$

14) $\frac{v-1}{4}\cdot\frac{4}{v^2-11v+10}$

15) $\frac{x^2-6x-7}{x+5}\cdot\frac{x+5}{x-7}$

16) $\frac{1}{a-6}\cdot\frac{8a+80}{8}$

17) $\frac{8k}{24k^2-40k}\div\frac{1}{15k-25}$

18) $\frac{p-8}{p^2-12p+32}\div\frac{1}{p-10}$

19) $(n-8)\cdot\frac{6}{10n-80}$

20) $\frac{x^2-7x+10}{x-2}\cdot\frac{x+10}{x^2-x-20}$

21) $\frac{4m+36}{m+9}\cdot\frac{m-5}{5m^2}$

22) $\frac{2r}{r+6}\div\frac{2r}{7r+42}$

23) $\frac{3x-6}{12x-24}(x+3)$

24) $\frac{2n^2-12n-54}{n+7}\div(2n+6)$

25) $\frac{b+2}{40b^2-24b}(5b-3)$

26) $\frac{21v^2+16v-16}{3v+4}\div\frac{35v-20}{v-9}$

27) $\frac{n-7}{6n-12}\cdot\frac{12-6n}{n^2-13n+42}$

28) $\frac{x^2+11x+24}{6x^3+18x^2}\cdot\frac{6x^3+6x^2}{x^2+5x-24}$

29) $\frac{27a+36}{9a+63}\div\frac{6a+8}{2}$

30) $\frac{k-7}{k^2-k-12}\cdot\frac{7k^2-28k}{8k^2-56k}$

31) $\frac{x^2-12x+32}{x^2-6x-16}\cdot\frac{7x^2+14x}{7x^2+21x}$

32) $\frac{9x^3+54x^2}{x^2+5x-14}\cdot\frac{x^2+5x-14}{10x^2}$

33) $(10m^2+100m)\cdot\frac{18m^3-36m^2}{20m^2-40m}$

34) $\frac{n-7}{n^2-2n-35}\div\frac{9n+54}{10n+50}$

35) $\frac{7p^2+25p+12}{6p+48}\cdot\frac{3p-8}{21p^2-44p-32}$

36) $\frac{7x^2-66x+80}{49x^2+7x-72}\div\frac{7x^2+39x-70}{49x^2+7x-72}$

37) $\frac{10b^2}{30b+20}\cdot\frac{30b+20}{2b^2+10b}$

38) $\frac{35n^2-12n-32}{49n^2-91n+40}\cdot\frac{7n^2+16n-15}{5n+4}$

39) $\frac{7r^2-53r-24}{7r+2}\div\frac{49r+21}{49r+14}$

40) $\frac{12x+24}{10x^2+34x+28}\cdot\frac{15x+21}{5}$

41) $\frac{x^2-1}{2x-4}\cdot\frac{x^2-4}{x^2-x-2}\div\frac{x^2+x-2}{3x-6}$

42) $\frac{a^3+b^3}{a^2+3\text{ab}+2b^2}\cdot\frac{3a-6b}{3a^2-3\text{ab}+3b^2}\div\frac{a^2-4b^2}{a+2b}$

43) $\frac{x^2+3x+9}{x^2+x-12}\cdot\frac{x^2+2x-8}{x^3-27}\div\frac{x^2-4}{x^2-6x+9}$

44) $\frac{x^2+3x-10}{x^2+6x+5}\cdot\frac{2x^2-x-3}{2x^2+x-6}\div\frac{8x+20}{6x+15}$

Answers - Multipy and Divide

1) $4x^2$

2) $\frac{14}{3}$

3) $\frac{63}{10n}$

4) $\frac{63}{10m}$

5) $\frac{3x^2}{2}$

6) $\frac{5p}{2}$

7) $5m$

8) $\frac{7}{10}$

9) $\frac{r-6}{r+10}$

10) $x+4$

11) $\frac{2}{3}$

12) $\frac{9}{b-5}$

13) $\frac{x-10}{7}$

14) $\frac{1}{v-10}$

15) $x+1$

16) $\frac{a+10}{a-6}$

17) 5

18) $\frac{p-10}{p-4}$

19) $\frac{3}{5}$

20) $\frac{x+10}{x+4}$

21) $\frac{4(m-5)}{5m^2}$

22) 7

23) $\frac{x+3}{4}$

24) $\frac{n-9}{n+7}$

25) $\frac{b+2}{8b}$

26) $\frac{v-9}{5}$

27) $-\frac{1}{n-6}$

28) $\frac{x+1}{x-3}$

29) $\frac{1}{a+7}$

30) $\frac{7}{8(k+3)}$

31) $\frac{x-4}{x+3}$

32) $\frac{9(x+6)}{10}$

33) $9m^2(m+10)$

34) $\frac{10}{9(n+6)}$

35) $\frac{p+3}{6(p+8)}$

36) $\frac{x-8}{x+7}$

37) $\frac{5b}{b+5}$

38) $n+3$

39) $r-8$

40) $\frac{18}{5}$

41) $\frac{3}{2}$

42) $\frac{1}{a+2b}$

43) $\frac{1}{x+2}$

44) $\frac{3(x-2)}{4(x+2)}$

4.3 Least Common Denominators

Objective: Idenfity the least common denominator and build up denominators to match this common denominator.

As with fractions, the least common denominator or LCD is very important to working with rational expressions. The process we use to find the LCD is based on the process used to find the LCD of intergers.

Example 1.

$$\begin{array}{rl} \text{Find the LCD of 8 and 6} & \text{Consider multiples of the larger number} \\ 8, 16, 24.... & \text{24 is the first multiple of 8 that is also divisible by 6} \\ 24 & \text{Our Solution} \end{array}$$

When finding the LCD of several monomials we first find the LCD of the coefficients, then use all variables and attach the highest exponent on each variable.

Example 2.

Find the LCD of $4x^2y^5$ and $6x^4y^3z^6$

$$\begin{array}{rl} & \text{First find the LCD of coefficients 4 and 6} \\ 12 & \text{12 is the LCD of 4 and 6} \\ x^4y^5z^6 & \text{Use all variables with highest exponents on each variable} \\ 12x^4y^5z^6 & \text{Our Solution} \end{array}$$

The same pattern can be used on polynomials that have more than one term. However, we must first factor each polynomial so we can identify all the factors to be used (attaching highest exponent if necessary).

Example 3.

$$\begin{array}{rl} \text{Find the LCD of } x^2+2x-3 \text{ and } x^2-x-12 & \text{Factor each polynomial} \\ (x-1)(x+3) \text{ and } (x-4)(x+3) & \text{LCD uses all unique factors} \\ (x-1)(x+3)(x-4) & \text{Our Solution} \end{array}$$

Notice we only used $(x+3)$ once in our LCD. This is because it only appears as a factor once in either polynomial. The only time we need to repeat a factor or use an exponent on a factor is if there are exponents when one of the polynomials is factored

Example 4.

Find the LCD of $x^2-10x+25$ and $x^2-14x+45$

	Factor each polynomial
$(x-5)^2$ and $(x-5)(x-9)$	LCD uses all unique factors with highest exponent
$(x-5)^2(x-9)$	Our Solution

The previous example could have also been done with factoring the first polynomial to $(x-5)(x-5)$. Then we would have used $(x-5)$ twice in the LCD because it showed up twice in one of the polynomials. However, it is the author's suggestion to use the exponents in factored form so as to use the same pattern (highest exponent) as used with monomials.

Once we know the LCD, our goal will be to build up fractions so they have matching denominators. In this lesson we will not be adding and subtracting fractions, just building them up to a common denominator. We can build up a fraction's denominator by multipliplying the numerator and denoinator by any factors that are not already in the denominator.

Example 5.

$\dfrac{5a}{3a^2b}=\dfrac{?}{6a^5b^3}$	Idenfity what factors we need to match denominators
$2a^3b^2$	$3\cdot 2=6$ and we need three more $a's$ and two more $b's$
$\dfrac{5a}{3a^2b}\left(\dfrac{2a^3b^2}{2a^3b^2}\right)$	Multiply numerator and denominator by this
$\dfrac{10a^4b^2}{6a^5b^3}$	Our Solution

Example 6.

$\dfrac{x-2}{x+4}=\dfrac{?}{x^2+7x+12}$	Factor to idenfity factors we need to match denominators
$(x+4)(x+3)$	
$(x+3)$	The missing factor
$\dfrac{x-2}{x+4}\left(\dfrac{x+3}{x+3}\right)$	Multiply numerator and denominator by missing factor,
	FOIL numerator
$\dfrac{x^2+x-6}{(x+4)(x+3)}$	Our Solution

As the above example illustrates, we will multiply out our numerators, but keep our denominators factored. The reason for this is to add and subtract fractions we will want to be able to combine like terms in the numerator, then when we reduce at the end we will want our denominators factored.

Once we know how to find the LCD and how to build up fractions to a desired denominator we can combine them together by finding a common denominator and building up those fractions.

Example 7.

Build up each fraction so they have a common denominator

$$\frac{5a}{4b^3c} \text{ and } \frac{3c}{6a^2b} \quad \text{First identify LCD}$$

$$12a^2b^3c \quad \text{Determine what factors each fraction is missing}$$

$$\text{First: } 3a^2 \quad \text{Second: } 2b^2c \quad \text{Multiply each fraction by missing factors}$$

$$\frac{5a}{4b^3c}\left(\frac{3a^2}{3a^2}\right) \text{ and } \frac{3c}{6a^2b}\left(\frac{2b^2c}{2b^2c}\right)$$

$$\frac{15a^3}{12a^2b^3c} \text{ and } \frac{6b^2c^2}{12a^2b^3c} \quad \text{Our Solution}$$

Example 8.

Build up each fraction so they have a common denominator

$$\frac{5x}{x^2-5x-6} \text{ and } \frac{x-2}{x^2+4x+3} \quad \text{Factor to find LCD}$$

$$(x-6)(x+1) \qquad (x+1)(x+3) \quad \text{Use factors to find LCD}$$

$$\text{LCD: } (x-6)(x+1)(x+3) \quad \text{Identify which factors are missing}$$

$$\text{First: } (x+3) \quad \text{Second: } (x-6) \quad \text{Multiply fractions by missing factors}$$

$$\frac{5x}{(x-6)(x+1)}\left(\frac{x+3}{x+3}\right) \text{ and } \frac{x-2}{(x+1)(x+3)}\left(\frac{x-6}{x-6}\right) \quad \text{Multiply numerators}$$

$$\frac{5x^2+15x}{(x-6)(x+1)(x+3)} \text{ and } \frac{x^2-8x+12}{(x-6)(x+1)(x+3)} \quad \text{Our Solution}$$

World View Note: When the Egyptians began working with fractions, they expressed all fractions as a sum of unit fraction. Rather than $\frac{4}{5}$, they would write the fraction as the sum, $\frac{1}{2}+\frac{1}{4}+\frac{1}{20}$. An interesting problem with this system is this is not a unique solution, $\frac{4}{5}$ is also equal to the sum $\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{10}$.

4.3 Practice - Least Common Denominator

Build up denominators.

1) $\frac{3}{8}=\frac{?}{48}$

2) $\frac{a}{5}=\frac{?}{5a}$

3) $\frac{a}{x}=\frac{?}{xy}$

4) $\frac{5}{2x^2}=\frac{?}{8x^3y}$

5) $\frac{2}{3a^3b^2c}=\frac{?}{9a^5b^2c^4}$

6) $\frac{4}{3a^5b^2c^4}=\frac{?}{9a^5b^2c^4}$

7) $\frac{2}{x+4}=\frac{?}{x^2-16}$

8) $\frac{x+1}{x-3}=\frac{?}{x^2-6x+9}$

9) $\frac{x-4}{x+2}=\frac{?}{x^2+5x+6}$

10) $\frac{x-6}{x+3}=\frac{?}{x^2-2x-15}$

Find Least Common Denominators

11) $2a^3, 6a^4b^2, 4a^3b^5$

12) $5x^2y, 25x^3y^5z$

13) $x^2-3x, x-3, x$

14) $4x-8, x-2, 4$

15) $x+2, x-4$

16) $x, x-7, x+1$

17) $x^2-25, x+5$

18) x^2-9, x^2-6x+9

19) x^2+3x+2, x^2+5x+6

20) $x^2-7x+10, x^2-2x-15, x^2+x-6$

Find LCD and build up each fraction

21) $\frac{3a}{5b^2}, \frac{2}{10a^3b}$

22) $\frac{3x}{x-4}, \frac{2}{x+2}$

23) $\frac{x+2}{x-3}, \frac{x-3}{x+2}$

24) $\frac{5}{x^2-6x}, \frac{2}{x}, \frac{-3}{x-6}$

25) $\frac{x}{x^2-16}, \frac{3x}{x^2-8x+16}$

26) $\frac{5x+1}{x^2-3x-10}, \frac{4}{x-5}$

27) $\frac{x+1}{x^2-36}, \frac{2x+3}{x^2+12x+36}$

28) $\frac{3x+1}{x^2-x-12}, \frac{2x}{x^2+4x+3}$

29) $\frac{4x}{x^2-x-6}, \frac{x+2}{x-3}$

30) $\frac{3x}{x^2-6x+8}, \frac{x-2}{x^2+x-20}, \frac{5}{x^2+3x-10}$

Answers - Least Common Denominators

1) 18

2) a^2

3) ay

4) $20xy$

5) $6a^2c^3$

6) 12

7) $2x-8$

8) x^2-2x-3

9) x^2-x-12

10) $x^2-11x+30$

11) $12a^4b^5$

12) $25x^3y^5z$

13) $x(x-3)$

14) $4(x-2)$

15) $(x+2)(x-4)$

16) $x(x-7)(x+1)$

17) $(x+5)(x-5)$

18) $(x-3)^2(x+3)$

19) $(x+1)(x+2)(x+3)$

20) $(x-2)(x-5)(x+3)$

21) $\frac{6a^4}{10a^3b^2}, \frac{2b}{10a^3b^2}$

22) $\frac{3x^2+6x}{(x-4)(x+2)}, \frac{2x-8}{(x-4)(x+2)}$

23) $\frac{x^2+4x+4}{(x-3)(x+2)}, \frac{x^2-6x+9}{(x-3)(x+2)}$

24) $\frac{5}{x(x-6)}, \frac{2x-12}{x(x-6)}, \frac{-3x}{x(x-6)}$

25) $\frac{x^2-4x}{(x-4)^2(x+4)}, \frac{3x^2+12x}{(x-4)^2(x+4)}$

26) $\frac{5x+1}{(x-5)(x+2)}, \frac{4x+8}{(x-5)(x+2)}$

27) $\frac{x^2+7x+6}{(x-6)(x+6)^2}, \frac{2x^2-9x-18}{(x-6)(x+6)^2}$

28) $\frac{3x^2+4x+1}{(x-4)(x+3)(x+1)}, \frac{2x^2-8x}{(x-4)(x+3)(x+1)}$

29) $\frac{4x}{(x-3)(x+2)}, \frac{x^2+4x+4}{(x-3)(x+2)}$

30) $\frac{3x^2+15x}{(x-4)(x-2)(x+5)}, \frac{x^2-4x+4}{(x-4)(x-2)(x+5)}, \frac{5x-20}{(x-4)(x-2)(x+5)}$

4.4 Add & Subtract

Objective: Add and subtract rational expressions with and without common denominators.

Adding and subtracting rational expressions is identical to adding and subtracting with integers. Recall that when adding with a common denominator we add the numerators and keep the denominator. This is the same process used with rational expressions. Remember to reduce, if possible, your final answer.

Example 1.

$$\frac{x-4}{x^2-2x-8}+\frac{x+8}{x^2-2x-8} \quad \text{Same denominator, add numerators, combine like terms}$$

$$\frac{2x+4}{x^2-2x-8} \quad \text{Factor numerator and denominator}$$

$$\frac{2(x+2)}{(x+2)(x-4)} \quad \text{Divide out } (x+2)$$

$$\frac{2}{x-4} \quad \text{Our Solution}$$

Subtraction with common denominator follows the same pattern, though the subtraction can cause problems if we are not careful with it. To avoid sign errors we will first distribute the subtraction through the numerator. Then we can treat it like an addition problem. This process is the same as "add the opposite" we saw when subtracting with negatives.

Example 2.

$$\frac{6x-12}{3x-6}-\frac{15x-6}{3x-6} \quad \text{Add the opposite of the second fraction (distribute negative)}$$

$$\frac{6x-12}{3x-6}+\frac{-15x+6}{3x-6}$$ Add numerators, combine like terms

$$\frac{-9x-6}{3x-6}$$ Factor numerator and denominator

$$\frac{-3(3x+2)}{3(x-2)}$$ Divide out common factor of 3

$$\frac{-(3x+2)}{x-2}$$ Our Solution

World View Note: The Rhind papyrus of Egypt from 1650 BC gives some of the earliest known symbols for addition and subtraction, a pair of legs walking in the direction one reads for addition, and a pair of legs walking in the opposite direction for subtraction..

When we don't have a common denominator we will have to find the least common denominator (LCD) and build up each fraction so the denominators match. The following example shows this process with integers.

Example 3.

$$\frac{5}{6}+\frac{1}{4}$$ The LCD is 12. Build up, multiply 6 by 2 and 4 by 3

$$\left(\frac{2}{2}\right)\frac{5}{6}+\frac{1}{4}\left(\frac{3}{3}\right)$$ Multiply

$$\frac{10}{12}+\frac{3}{12}$$ Add numerators

$$\frac{13}{12}$$ Our Solution

The same process is used with variables.

Example 4.

$$\frac{7a}{3a^2b}+\frac{4b}{6ab^4}$$ The LCD is $6a^2b^4$. We will then build up each fraction

$$\left(\frac{2b^3}{2b^3}\right)\frac{7a}{3a^2b}+\frac{4b}{6ab^4}\left(\frac{a}{a}\right) \quad \text{Multiply first fraction by } 2b^3 \text{ and second by } a$$

$$\frac{14ab^3}{6a^2b^4}+\frac{4ab}{6a^2b^4} \quad \text{Add numerators, no like terms to combine}$$

$$\frac{14ab^3+4ab}{6a^2b^4} \quad \text{Factor numerator}$$

$$\frac{2ab(7b^3+2)}{6a^2b^4} \quad \text{Reduce, dividing out factors } 2,\ a, \text{ and } b$$

$$\frac{7b^3+2}{3ab^3} \quad \text{Our Solution}$$

The same process can be used for subtraction, we will simply add the first step of adding the opposite.

Example 5.

$$\frac{4}{5a}-\frac{7b}{4a^2} \quad \text{Add the opposite}$$

$$\frac{4}{5a}+\frac{-7b}{4a^2} \quad \text{LCD is } 20a^2\text{. Build up denominators}$$

$$\left(\frac{4a}{4a}\right)\frac{4}{5a}+\frac{-7b}{4a^2}\left(\frac{5}{5}\right) \quad \text{Multiply first fraction by } 4a\text{, second by } 5$$

$$\frac{16a-35b}{20a^2} \quad \text{Our Solution}$$

If our denominators have more than one term in them we will need to factor first to find the LCD. Then we build up each denominator using the factors that are missing on each fraction.

Example 6.

$$\frac{6}{8a+4}+\frac{3a}{8} \quad \text{Factor denominators to find LCD}$$

$$4(2a+1) \quad 8 \qquad \text{LCD is } 8(2a+1)\text{, build up each fraction}$$

$$\left(\frac{2}{2}\right)\frac{6}{4(2a+1)}+\frac{3a}{8}\left(\frac{2a+1}{2a+1}\right) \qquad \text{Multiply first fraction by 2, second by } 2a+1$$

$$\frac{12}{8(2a+1)}+\frac{6a^2+3a}{8(2a+1)} \qquad \text{Add numerators}$$

$$\frac{6a^2+3a+12}{8(2a+1)} \qquad \text{Our Solution}$$

With subtraction remember to add the opposite.

Example 7.

$$\frac{x+1}{x-4}-\frac{x+1}{x^2-7x+12} \qquad \text{Add the opposite (distribute negative)}$$

$$\frac{x+1}{x-4}+\frac{-x-1}{x^2-7x+12} \qquad \text{Factor denominators to find LCD}$$

$$x-4 \quad (x-4)(x-3) \qquad \text{LCD is } (x-4)(x-3)\text{, build up each fraction}$$

$$\left(\frac{x-3}{x-3}\right)\frac{x+1}{x-4}+\frac{-x-1}{x^2-7x+12} \qquad \text{Only first fraction needs to be multiplied by } x-3$$

$$\frac{x^2-2x-3}{(x-3)(x-4)}+\frac{-x-1}{(x-3)(x-4)} \qquad \text{Add numerators, combine like terms}$$

$$\frac{x^2-3x-4}{(x-3)(x-4)} \qquad \text{Factor numerator}$$

$$\frac{(x-4)(x+1)}{(x-3)(x-4)} \qquad \text{Divide out } x-4 \text{ factor}$$

$$\frac{x+1}{x-3} \qquad \text{Our Solution}$$

4.4 Practice - Add and Subtract

Add or subtract the rational expressions. Simplify your answers whenever possible.

1) $\frac{2}{a+3}+\frac{4}{a+3}$

2) $\frac{x^2}{x-2}-\frac{6x-8}{x-2}$

3) $\frac{t^2+4t}{t-1}+\frac{2t-7}{t-1}$

4) $\frac{a^2+3a}{a^2+5a-6}-\frac{4}{a^2+5a-6}$

5) $\frac{2x^2+3}{x^2-6x+5}-\frac{x^2-5x+9}{x^2-6x+5}$

6) $\frac{3}{x}+\frac{4}{x^2}$

7) $\frac{5}{6r}-\frac{5}{8r}$

8) $\frac{7}{xy^2}+\frac{3}{x^2y}$

9) $\frac{8}{9t^3}+\frac{5}{6t^2}$

10) $\frac{x+5}{8}+\frac{x-3}{12}$

11) $\frac{a+2}{2}-\frac{a-4}{4}$

12) $\frac{2a-1}{3a^2}+\frac{5a+1}{9a}$

13) $\frac{x-1}{4x}-\frac{2x+3}{x}$

14) $\frac{2c-d}{c^2d}-\frac{c+d}{cd^2}$

15) $\frac{5x+3y}{2x^2y}-\frac{3x+4y}{xy^2}$

16) $\frac{2}{x-1}+\frac{2}{x+1}$

17) $\frac{2z}{z-1}-\frac{3z}{z+1}$

18) $\frac{2}{x-5}+\frac{3}{4x}$

19) $\frac{8}{x^2-4}-\frac{3}{x+2}$

20) $\frac{4x}{x^2-25}+\frac{x}{x+5}$

21) $\frac{t}{t-3}-\frac{5}{4t-12}$

22) $\frac{2}{x+3}+\frac{4}{(x+3)^2}$

23) $\frac{2}{5x^2+5x}-\frac{4}{3x+3}$

24) $\frac{3a}{4a-20}+\frac{9a}{6a-30}$

25) $\frac{t}{y-t}-\frac{y}{y+t}$

26) $\frac{x}{x-5}+\frac{x-5}{x}$

27) $\frac{x}{x^2+5x+6}-\frac{2}{x^2+3x+2}$

28) $\frac{2x}{x^2-1}-\frac{3}{x^2+5x+4}$

29) $\frac{x}{x^2+15x+56}-\frac{7}{x^2+13x+42}$

30) $\frac{2x}{x^2-9}+\frac{5}{x^2+x-6}$

31) $\frac{5x}{x^2-x-6}-\frac{18}{x^2-9}$

32) $\frac{4x}{x^2-2x-3}-\frac{3}{x^2-5x+6}$

33) $\frac{2x}{x^2-1}-\frac{4}{x^2+2x-3}$

34) $\frac{x-1}{x^2+3x+2}+\frac{x+5}{x^2+4x+3}$

35) $\frac{x+1}{x^2-2x-35}+\frac{x+6}{x^2+7x+10}$

36) $\frac{3x+2}{3x+6}+\frac{x}{4-x^2}$

37) $\frac{4-a^2}{a^2-9}-\frac{a-2}{3-a}$

38) $\frac{4y}{y^2-1}-\frac{2}{y}-\frac{2}{y+1}$

39) $\frac{2z}{1-2z}+\frac{3z}{2z+1}-\frac{3}{4z^2-1}$

40) $\frac{2r}{r^2-s^2}+\frac{1}{r+s}-\frac{1}{r-s}$

41) $\frac{2x-3}{x^2+3x+2}+\frac{3x-1}{x^2+5x+6}$

42) $\frac{x+2}{x^2-4x+3}+\frac{4x+5}{x^2+4x-5}$

43) $\frac{2x+7}{x^2-2x-3}-\frac{3x-2}{x^2+6x+5}$

44) $\frac{3x-8}{x^2+6x+8}+\frac{2x-3}{x^2+3x+2}$

4.4

Answers - Add and Subtract

1) $\frac{6}{a+3}$

2) $x-4$

3) $t+7$

4) $\frac{a+4}{a+6}$

5) $\frac{x+6}{x-5}$

6) $\frac{3x+4}{x^2}$

7) $\frac{5}{24r}$

8) $\frac{7x+3y}{x^2y^2}$

9) $\frac{15t+16}{18t^3}$

10) $\frac{5x+9}{24}$

11) $\frac{a+8}{4}$

12) $\frac{5a^2+7a-3}{9a^2}$

13) $\frac{-7x-13}{4x}$

14) $\frac{-c^2+cd-d^2}{c^2d^2}$

15) $\frac{3y^2-3xy-6x^2}{2x^2y^2}$

16) $\frac{4x}{x^2-1}$

17) $\frac{-z^2+5z}{z^2-1}$

18) $\frac{11x+15}{4x(x+5)}$

19) $\frac{14-3x}{x^2-4}$

20) $\frac{x^2-x}{x^2-25}$

21) $\frac{4t-5}{4(t-3)}$

22) $\frac{2x+10}{(x+3)^2}$

23) $\frac{6-20x}{15x(x+1)}$

24) $\frac{9a}{4(a-5)}$

25) $\frac{t^2+2ty-y^2}{y^2-t^2}$

26) $\frac{2x^2-10x+25}{x(x-5)}$

27) $\frac{x-3}{(x+3)(x+1)}$

28) $\frac{2x+3}{(x-1)(x+4)}$

29) $\frac{x-8}{(x+8)(x+6)}$

30) $\frac{2x-5}{(x-3)(x-2)}$

31) $\frac{5x+12}{x^2+5x+6}$

32) $\frac{4x+1}{(x+1)(x-2)}$

33) $\frac{2x+4}{x^2+4x+3}$

34) $\frac{2x+7}{x^2+5x+6}$

35) $\frac{2x-8}{x^2-5x-14}$

36) $\frac{-3x^2+7x+4}{3(x+2)(2-x)}$

37) $\frac{a-2}{a^2-9}$

38) $\frac{2}{y^2-y}$

39) $\frac{z-3}{2z-1}$

40) $\frac{2}{r+s}$

41) $\frac{5(x-1)}{(x+1)(x+3)}$

42) $\frac{5x+5}{x^2+2x-15}$

43) $\frac{-(x-29)}{(x-3)(x+5)}$

44) $\frac{5x-10}{x^2+5x+4}$